29. **Implicit differentiation**

29.1. *The recipe*

Recall that an implicitely defined function is a function $y = f(x)$ which is defined by an equation of the form

$$F(x, y) = 0.$$

We call this equation the ***defining equation*** for the function $y = f(x)$. To find $y = f(x)$ for a given value of x you must solve the defining equation $F(x, y) = 0$ for y.

Here is a recipe for computing the derivative of an implicitely defined function.

(1) Differentiate the equation $F(x, y) = 0$; you may need the chain rule to deal with the occurences of y in $F(x, y)$;

(2) You can rearrange the terms in the result of step 1 so as to get an equation of the form

$$G(x, y)\frac{dy}{dx} + H(x, y) = 0,$$

where G and H are expressions containing x and y but not the derivative.

(3) Solve the equation in step 2 for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = -\frac{H(x, y)}{G(x, y)} \tag{28}$$

(4) If you also have an explicit description of the function (i.e. a formula expressing $y = f(x)$ in terms of x) then you can substitute $y = f(x)$ in the expression (28) to get a formula for dy/dx in terms of x only.

Often no explicit formula for y is available and you can't take this last step. In that case (28) is as far as you can go.

Observe that by following this procedure you will get a formula for the derivative $\frac{dy}{dx}$ which contains both x and y.

29.2. *Dealing with equations of the form* $F_1(x, y) = F_2(x, y)$

If the implicit definition of the function is not of the form $F(x, y) = 0$ but rather of the form $F_1(x, y) = F_2(x, y)$ then you move all terms to the left hand side, and proceed as above. E.g. to deal with a function $y = f(x)$ which satisfies

$$y^2 + x = xy$$

you rewrite this equation as

$$y^2 + x - xy = 0$$

and set $F(x, y) = y^2 + x - xy$.

29.3. *Example – Derivative of* $\sqrt[4]{1 - x^4}$

Consider the function

$$f(x) = \sqrt[4]{1 - x^4}, \quad -1 \le x \le 1.$$

We will compute its derivative in two ways: first the direct method, and then using the method f implicit differentiation (i.e. the recipe above).

The direct approach goes like this:

$$\begin{aligned} f'(x) &= \frac{d\left(1-x^4\right)^{1/4}}{dx} \\ &= \tfrac{1}{4}\left(1-x^4\right)^{-3/4}\frac{d(1-x^4)}{dx} \\ &= \tfrac{1}{4}\left(1-x^4\right)^{-3/4}\left(-4x^3\right) \\ &= -\frac{x^3}{\left(1-x^4\right)^{3/4}} \end{aligned}$$

To find the derivative using implicit differentiation we must first find a nice implicit description of the function. For instance, we could decide to get rid of all roots or fractional exponents in the function and point out that $y = \sqrt[4]{1-x^4}$ satisfies the equation $y^4 = 1-x^4$. So our implicit description of the function $y = f(x) = \sqrt[4]{1-x^4}$ is

$$x^4 + y^4 - 1 = 0.$$

$F(x, y) = x^4 + y^4 - 1$

Differentiate both sides with respect to x (and remember that $y = f(x)$, so y here is a function of x), and you get

$$\frac{dx^4}{dx} + \frac{dy^4}{dx} = \frac{d1}{dx} \implies 4x^3 + 4y^3\frac{dy}{dx} = 0.$$

$G(x, y) = 4y^3$
$H(x, y) = 4x^3$

This last equation can be solved for dy/dx:

$$\frac{dy}{dx} = -\frac{x^3}{y^3}.$$

This is a nice and short form of the derivative, but it contains y as well as x. To express dy/dx in terms of x only, and remove the y dependency we use $y = \sqrt[4]{1-x^4}$. The result is

$$f'(x) = \frac{dy}{dx} = -\frac{x^3}{y^3} = -\frac{x^3}{\left(1-x^4\right)^{3/4}}.$$

29.4. *Another example*

Let f be a function defined by

$$y = f(x) \iff 2y + \sin y = x, \text{ i.e. } 2y + \sin y - x = 0.$$

For instance, if $x = 2\pi$ then $y = \pi$, i.e. $f(2\pi) = \pi$.

To find the derivative dy/dx we differentiate the defining equation

$$\frac{d(2y + \sin y - x)}{dx} = \frac{d0}{dx} \implies 2\frac{dy}{dx} + \cos y\frac{dy}{dx} - \frac{dx}{dx} = 0 \implies (2 + \cos y)\frac{dy}{dx} - 1 = 0.$$

Solve for $\frac{dy}{dx}$ and you get

$$f'(x) = \frac{1}{2 + \cos y} = \frac{1}{2 + \cos f(x)}.$$

If we were asked to find $f'(2\pi)$ then, since we know $f(2\pi) = \pi$, we could answer

$$f'(2\pi) = \frac{1}{2 + \cos\pi} = \frac{1}{2-1} = 1.$$

If we were asked $f'(\pi/2)$, then all we would be able to say is

$$f'(\pi/2) = \frac{1}{2 + \cos f(\pi/2)}.$$

To say more we would first have to find $y = f(\pi/2)$, which one does by solving

$$2y + \sin y = \frac{\pi}{2}.$$

29.5. *Derivatives of Arc Sine and Arc Tangent*

Recall that

$$y = \arcsin x \iff x = \sin y \text{ and } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2},$$

and

$$y = \arctan x \iff x = \tan y \text{ and } -\tfrac{\pi}{2} < y < \tfrac{\pi}{2}.$$

Theorem 29.1.

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$$

Proof. If $y = \arcsin x$ then $x = \sin y$. Differentiate this relation

$$\frac{dx}{dx} = \frac{d \sin y}{dx}$$

and apply the chain rule. You get

$$1 = \bigl(\cos y\bigr)\,\frac{dy}{dx},$$

and hence

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

How do we get rid of the y on the right hand side? We know $x = \sin y$, and also $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Therefore

$$\sin^2 y + \cos^2 y = 1 \implies \cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}.$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ we know that $\cos y \ge 0$, so we must choose the positive square root. This leaves us with $\cos y = \sqrt{1-x^2}$, and hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The derivative of $\arctan x$ is found in the same way, and you should really do this yourself. □

Exercises on implicit differentiation

<u>*29.1*</u> – Find the derivative $f'(x)$ if $y = f(x)$ satisfies

(i) $xy = \dfrac{\pi}{6}$	**(ii)** $\sin(xy) = \frac{1}{2}$
(iii) $\dfrac{xy}{x+y} = 1$	**(iv)** $x + y = xy$
(v) $(y-1)^2 + x = 0$	**(vi)** $(y+1)^2 + y - x = 0$
(vii) $(y-x)^2 + x = 0$	**(viii)** $(y+x)^2 + 2y - x = 0$
(ix) $\bigl(y^2-1\bigr)^2 + x = 0$	**(x)** $\bigl(y^2+1\bigr)^2 - x = 0$
(xi) $x^3 + xy + y^3 = 3$	**(xii)** $\sin x + \sin y = 1$
(xiii) $\sin x + xy + y^5 = \pi$	**(xiv)** $\tan x + \tan y = 1$

For each of these problems state what the expressions $F(x,y)$, $G(x,y)$ and $H(x,y)$ from the recipe in the beginning of this section are.

If you can find an explicit description of the function $y = f(x)$, say what it is.

29.2 – For each of the following explicitly defined functions find an implicit definition which does not involve taking roots. Then use this description to find the derivative dy/dx.

(i) $y = f(x) = \sqrt{1-x}$ **(ii)** $y = f(x) = \sqrt[4]{x+x^2}$

(iii) $y = f(x) = \sqrt{1-\sqrt{x}}$ **(iv)** $y = f(x) = \sqrt[4]{x-\sqrt{x}}$

(v) $y = f(x) = \sqrt[3]{\sqrt{2x+1}-x^2}$ **(vi)** $y = f(x) = \sqrt[4]{x+x^2}$

(vii) $y = f(x) = \sqrt[3]{x-\sqrt{2x+1}}$ **(viii)** $y = f(x) = \sqrt[4]{\sqrt[3]{x}}$

29.3 (Inverse trig review) – Simplify the following expressions, and indicate for which values of x (or θ, or ...) your simplification is valid. In case of doubt, try plotting the function on a graphing calculator.

(i) $\sin \arcsin x$ **(ii)** $\tan \arctan z$

(iii) $\cos \arcsin x$ **(iv)** $\tan \arcsin \theta$

(v) $\arctan(\tan \theta)$ **(vi)** $\arcsin(\sin \theta)$

(vii) $\cot \arctan x$ **(viii)** $\cot \arcsin x$

29.4 – Now that you know the derivatives of arcsin and arctan, you can find the derivatives of the following functions. What are they?

(i) $f(x) = \arcsin(2x)$ **(ii)** $f(x) = \arcsin \sqrt{x}$

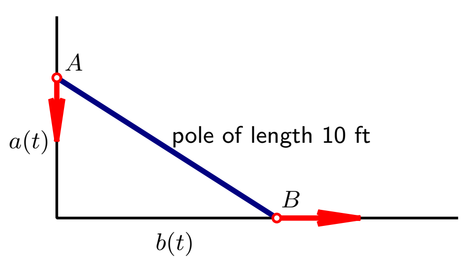
(iii) $f(x) = \arctan(\sin x)$ **(iv)** $f(x) = \sin \arctan x$

(v) $f(x) = \big(\arcsin x\big)^2$ **(vi)** $f(x) = \dfrac{1}{1+(\arctan x)^2}$

(vii) $f(x) = \sqrt{1-(\arcsin x)^2}$ **(viii)** $f(x) = \dfrac{\arctan x}{\arcsin x}$

Exercises on rates of change

29.5 – A 10 foot long pole has one end (B) on the floor and another (A) against a wall.



If the bottom of the pole is 8 feet away from the wall, and if it is sliding away from the wall at 7 feet per second, then with what speed is the top (A) going down?

29.6 – A pole 10 feet long rests against a vertical wall. If the bottom of the pole slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the pole and the wall changing when the angle is $\pi/4$ radians?

29.7 – A pole 13 meters long is leaning against a wall. The bottom of the pole is pulled along the ground away from the wall at the rate of 2 m/s. How fast is its height on the wall decreasing when the foot of the pole is 5 m away from the wall?

29.8 – A television camera is positioned 4000 ft from the base of a rocket launching pad. A rocket rises vertically and its speed is 600 ft/s when it has risen 3000 feet.

(**i**) How fast is the distance from the television camera to the rocket changing at that moment?

(**ii**) How fast is the camera's angle of elevation changing at that same moment? (Assume that the television camera points toward the rocket.)

29.9 – An isosceles triangle is changing its shape: the lengths of the two equal sides remain fixed at 2 inch, but the angle $\theta(t)$ between them changes.

Let $A(t)$ be the area of the triangle at time t. If the area increases at a constant rate of 0.5inch^2/sec, then how fast is the angle increasing or decreasing when $\theta = 60°$?

29.10 – A point P is moving in the first quadrant of the plane. Its motion is parallel to the x-axis; its distance to the x-axis is always 10 (feet). Its velocity is 3 feet per second to the left. We write θ for the angle between the positive x-axis and the line segment from the origin to P.

(**i**) Make a drawing of the point P.

(**ii**) Where is the point when $\theta = \pi/3$?

(**iii**) Compute the rate of change of the angle θ at the moment that $\theta = \frac{\pi}{3}$.

29.11 – The point Q is moving on the line $y = x$ with velocity 3 m/sec. Find the rate of change of the following quantities at the moment in which Q is at the point $(1, 1)$:

(**i**) the distance from Q to the origin,

(**ii**) the distance from Q to the point $R(2, 0)$,

(**iii**) the angle $\angle ORQ$ where R is again the point $R(2, 0)$.

29.12 – A point P is sliding on the parabola with equation $y = x^2$. Its x-coordinate is increasing at a constant rate of 2 feet/minute.

Find the rate of change of the following quantities at the moment that P is at $(3, 9)$:

(**i**) the distance from P to the origin,

(**ii**) the area of the rectangle whose lower left corner is the origin and whose upper right corner is P,

(**iii**) the slope of the tangent to the parabola at Q,

(**iv**) the angle $\angle OPQ$ where Q is the point $(0, 3)$.

29.13 – A certain amount of gas is trapped in a cylinder with a piston. The ***ideal gas law*** from thermodynamics says that if the cylinder is not heated, and if the piston moves slowly, then one has

$$pV = CT$$

where p is the pressure in the gas, V is its volume, T its temperature (in degrees Kelvin) and C is a constant depending on the amount of gas trapped in the cylinder.

(**i**) If the pressure is 10psi (pounds per square inch), if the volume is 25inch^3, and if the piston is moving so that the gas volume is expanding at a rate of 2inch^3 per minute, then what is the rate of change of the pressure?

(**ii**) The ideal gas law turns out to be only approximately true. A more accurate description of gases is given by ***van der Waals' equation of state***, which says that

$$\left(p + \frac{a}{V^2}\right)(V - b) = C$$

where a, b, C are constants depending on the temperature and the amount and type of gas in the cylinder.

Suppose that the cylinder contains fictitious gas for which one has $a = 12$ and $b = 3$. Suppose that at some moment the volume of gas is 12in^3, the pressure is 25psi and suppose the gas is expanding at 2 inch^3 per minute. Then how fast is the pressure changing?