41. **Optimization Problems**

Often a problem can be phrased as

For which value of x in the interval $a \le x \le b$ is $f(x)$ the largest?

In other words you are given a function f on an interval $[a, b]$ and you must find all global maxima of f on this interval.

If the function is continuous then according to theorem 36.1 there always is at least one x in the interval $[a, b]$ which maximizes $f(x)$.

If f is differentiable then we know what to do: any local maximum is either a stationary point or one of the end points a and b. Therefore you can find the global maxima by following this recipe:

(1) Find all stationary points of f;
(2) Compute $f(x)$ at each stationary point you found in step (1);
(3) Compute $f(a)$ and $f(b)$;
(4) The global maxima are those stationary- or endpoints from steps (2) and (3) which have the largest function value.

Usually there is only one global maximum, but sometimes there can be more.

If you have to *minimize* rather than *maximize* a function, then you must look for global minima. The same recipe works (of course you should look for the smallest function value instead of the largest in step 4.)

The difficulty in optimization problems frequently lies not with the calculus part, but rather with setting up the problem. Choosing which quantity to call x and finding the function f is half the job.

41.1. *Example – The rectangle with largest area and given perimeter*

Which rectangle has the largest area, among all those rectangles for which the total length of the sides is 1?

Solution. If the sides of the rectangle have lengths x and y, then the total length of the sides is

$$L = x + x + y + y = 2(x + y)$$

and the area of the rectangle is

$$A = xy.$$

So are asked to find the largest possible value of $A = xy$ provided $2(x + y) = 1$. The lengths of the sides can also not be negative, so x and y must satisfy $x \ge 0$, $y \ge 0$.

We now want to turn this problem into a question of the form "maximize a function over some interval." The quantity which we are asked to maximize is A, but it depends on two variables x and y instead of just one variable. However, the variables x and y are not independent since we are only allowed to consider rectangles with $L = 1$. From this equation we get

$$L = 1 \implies y = \tfrac{1}{2} - x.$$

Hence we must find the maximum of the quantity

$$A = xy = x\big(\tfrac{1}{2} - x\big)$$

The values of x which we are allowed to consider are only limited by the requirements $x \ge 0$ and $y \ge 0$, i.e. $x \le \frac{1}{2}$. So we end up with this problem:

Find the maximum of the function $f(x) = x\big(\frac{1}{2} - x\big)$ on the interval $0 \le x \le \frac{1}{2}$.

Before we start computing anything we note that the function f is a polynomial so that it is differentiable, and hence continuous, and also that the interval $0 \le x \le \frac{1}{2}$ is closed. Therefore the theory guarantees that there is a maximum and our recipe will show us where it is.

The derivative is given by

$$f'(x) = \tfrac{1}{2} - 2x,$$

and hence the only stationary point is $x = \frac{1}{4}$. The function value at this point is

$$f\tfrac{1}{4}) = \tfrac{1}{4}(\tfrac{1}{2} - \tfrac{1}{4}) = \tfrac{1}{16}.$$

At the endpoints one has $x = 0$ or $x = \frac{1}{2}$, which corresponds to a rectangle one of whose sides has length zero. The area of such rectangles is zero, and so this is not the maximal value we are looking for.

We conclude that the largest area is attained by the rectangle whose sides have lengths

$$x = \tfrac{1}{4}, \text{ and } y = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4},$$

i.e. by a square with sides $\frac{1}{4}$.

41.2. *Exercises*

41.1 – By definition, the perimeter of a rectangle is the sum of the lengths of its four sides. Which rectangle, of all those whose perimeter is 1, has the smallest area? Which one has the largest area?

41.2 – Which rectangle of area 100in^2 minimizes its height plus two times its length?

41.3 – You have 1 yard of string from which you make a circular wedge with radius R and opening angle θ. Which choice of θ and R will give you the wedge with the largest area? Which choice leads to the smallest area?

[A circular wedge is the figure consisting of two radii of a circle and the arc connecting them. So the yard of string is used to form the two radii and the arc.]

41.4 – A rounded rectangle is a rectangle of width W and height H, with two half circles on the left and right sides glued on (so the circles have diameter H.) Which rounded rectangle with area 10in^2 has the smallest perimeter?

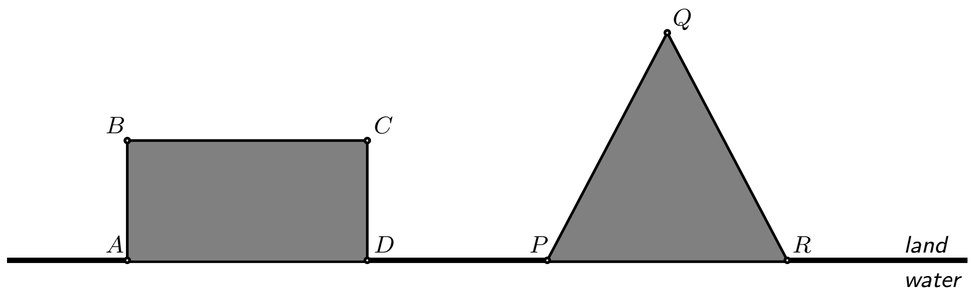
41.5 – (**a**) You have a sheet of metal with area $100\ \text{in}^2$ from which you are to make a cylindrical soup can. If r is the radius of the can and h its height, then which h and r will give you the can with the largest volume?

(**b**) If instead of making a plain cylinder you replaced the flat top and bottom of the cylinder with two spherical caps, then (using the same 100in^2 os sheet metal), then which choice of radius and height of the cylinder give you the container with the largest volume?

(**c**) Suppose you only replace the top of the cylinder with a spherical cap, and leave the bottom flat, then which choice of height and radius of the cylinder result in the largest volume?

41.6 – A triangle has one vertex at the origin $O(0,0)$, another at the point $A(2a,0)$ and the third at $(a, a/(1+a^3))$. What are the largest and smallest areas this triangle can have if $0 \le a < \infty$?

41.7 – According to tradition Dido was the founder and first Queen of Carthage. When she arrived on the north coast of Africa (∼800BC) the locals allowed her to take as much land as could be enclosed with the hide of one ox. She cut the hide into thin strips and put these together to form a length of 100 yards[5].



(**a**) If Dido wanted a rectangular region, then how wide should she choose it to enclose as much area as possible (the coastal edge of the boundary doesn't count, so in this problem the length $AB + BC + CD$ is 100 yards.)

(**b**) If Dido chose a region in the shape of an isosceles triangle PQR, then how wide should she make it to maximize its area (again, don't include the coast in the perimiter: $PQ+QR$ is 100 yards long, and $PQ = QR$.)

41.8 – The product of two numbers x, y is 16. We know $x \geq 1$ and $y \geq 1$. What is the greatest possible sum of the two numbers?

41.9 – What are the smallest and largest values that $(\sin x)(\sin y)$ can have if $x + y = \pi$ and if x and y are both nonnegative?

41.10 – What are the smallest and largest values that $(\cos x)(\cos y)$ can have if $x+y = \frac{\pi}{2}$ and if x and y are both nonnegative?

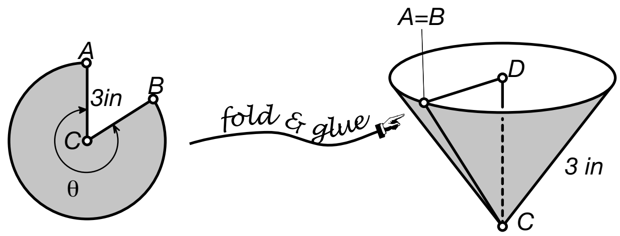
41.11 – (**a**) What are the smallest and largest values that $\tan x+\tan y$ can have if $x+y = \frac{\pi}{2}$ and if x and y are both nonnegative?

(**b**) What are the smallest and largest values that $\tan x + 2\tan y$ can have if $x + y = \frac{\pi}{2}$ and if x and y are both nonnegative?

41.12 – The cost per hour of fuel to run a locomotive is $v^2/25$ dollars, where v is speed (in miles per hour), and other costs are \$100 per hour regardless of speed. What is the speed that minimizes cost per mile ?

41.13 – Josh is in need of coffee. He has a circular filter with 3 inch radius. He cuts out a wedge and glues the two edges AC and BC together to make a conical filter to hold the ground coffee. The volume V of the coffee cone depends the angle θ of the piece of filter paper Josh made.

[5] I made that number up. For the rest start at `http://en.wikipedia.org/wiki/Dido`



(**a**) Find the volume in terms of the angle θ. (Hint: how long is the circular arc AB on the left? How long is the circular top of the cone on the right? If you know that you can find the radius $AD = BD$ of the top of the cone, and also the height CD of the cone.)

(**b**) Which angle θ maximizes the volume V?