PRACTICE PROBLEMS FOR THE FINAL EXAM

ε–δ problems

(1) If you know that $|h| < \delta$ then how large can $|-2h + h^2|$ be? *Your answer should be of the form "$|-2h+h^2| \leq$ [some expression which contains δ but not h]."*

(2) (a) If you know that $|h| < \delta$ then how large can $\left|\dfrac{h}{1+h}\right|$ be?

(b) If you know that $|h| < \delta$ and also that $\delta < \frac{1}{4}$, then how large can $\left|\dfrac{h}{1+h}\right|$ be?

(c) Can you find a constant K such that

$$|h| < \delta \implies \left|\frac{h}{1+h}\right| \leq K\delta \text{ ??}$$

You don't have to do this for all δ, but only for "small" δ, i.e. you may assume that δ is smaller than some constant of your choice (like 1, $\frac{1}{3}$, ... you must say which constant you choose.)

(3) Exactly the same question as (2), except change $h/(1+h)$ into $h/(1+2h)$.

(4) Suppose you know that $|h| < \delta$. Find a constant K so that

$$|-3h + h^2 - \tfrac{1}{2}h^4| \leq K\delta.$$

You may assume that δ is smaller than some constant of your choice (like 1, $\frac{1}{3}$, ... you must say which constant you choose.)

(5) Let

$$f(x) = x^2 + 2x.$$

(a) If $|x-2| < \delta$, then how large can $|f(x) - f(2)|$ be? More specifically, find a number $K > 0$ such that $|f(x) - f(2)| < K\delta$ holds whenever $|x-2| < \delta$.

If you need to, you may assume that δ is smaller than some given number (like 1, or $\frac{1}{10}$), but you must specify this number.

(b) Using your answer to (a) show that f is continuous at $x = 2$.

(6) Let

$$f(x) = \frac{1}{x} - 4x^2.$$

(a) If $|x - \frac{1}{2}| < \delta$, then how large can $|f(x) - 1|$ be? More specifically, find a number $K > 0$ such that $|f(x) - 1| < K\delta$ holds whenever $|x - \frac{1}{2}| < \delta$.

If you need to, you may assume that δ is smaller than some given number (like 1, or $\frac{1}{10}$), but you must specify this number.

(b) Using your answer to (a) show that $\lim_{x\to 1/2} f(x) = 1$.

Using limit properties

(see Chapter III, §16.4)

(7) Use the limit properties to show that

$$\lim_{x\to 3} \frac{1+x}{x-3}$$

does not exist.

(8) Use the limit properties to show that

$$\lim_{x\to\infty} x + \frac{1}{x}$$

does not exist.

(9) Use the limit properties to show that

$$\lim_{x\to 0} x + \frac{1}{x}$$

does not exist.

(10) Use the limit properties to show that

$$\lim_{x\to\pi} \frac{2-x}{\sin x}$$

does not exist.

Derivatives from the definition

(11) Use the definition of the derivative to show that

$$f(x) = (1+x)^2 \implies f'(x) = 2 + 2x.$$

(12) Assuming that

$$\lim_{h\to 0} \bigl(3^h - 1\bigr)/h = L$$

exists, prove directly from the definition of the derivative that

$$f(x) = 3^x \implies f'(x) = L3^x$$

Use what you know about e and ln to find a formula for L.

Implicit functions and their derivatives

(13) A problem like the one on the 1st midterm ; see §29 of the notes.

Derivatives & Limits involving exponentials, logs and inverse trig functions

(14) The hyperbolic cosine function is defined by

$$\cosh x \stackrel{\text{def}}{=} \frac{e^x + e^{-x}}{2}.$$

To see its graph upside down type "Gateway to the West" in `http://images.google.com`, but here is the math problem:

Show that

$$\frac{d}{dx}\bigl\{\arctan e^x\bigr\} = \frac{A}{\cosh x}$$

for some constant A, and find A.

(15) Find the first and second derivatives of $f(x) = x^x$ and of $g(x) = x^{2/\ln x}$.

(16) Compute

$$\lim_{x\searrow 0} \frac{\ln x}{x}, \quad \lim_{x\searrow 0} x^x, \quad \lim_{x\searrow 0} x^{1/\ln x}.$$

Max-min and Graph Sketching

(17) Sketch the graph of $f(x) = x^2 e^{-x^2}$.

(18) What is the largest value that $f(x) = \dfrac{x^2}{A + x^4}$ can have if $0 \le x < \infty$? (A is a positive constant.)

(19) For more practice redo problems 40.4–40.9, 41.1–41.12, and §49.1–49.12 from the notes.

Integration – Antiderivatives and the Fundamental Theorem

There will be no questions of the form "compute the following integral." All integrals that you will be asked to do will appear in problems asking you for (1) an area, (2) a volume (of a solid of rotation) or (3) an arclength.

Other questions involving integrals will require you to know the fundamental theorem, as in the following two problems (see also §55 of the notes):

(20) If

$$f(x) = \int_x^{x^2} \ln(1+x)\, dx$$

then compute $f'(1)$ and $f''(1)$.

(21) Compute

$$\frac{d}{dx}\int_x^1 \bigl(1 + t^2 + t^4\bigr)^3\, dt$$

(22) Does the graph of
$$F(x) = \int_0^x \frac{dt}{(1+t^2)^2}$$
have an inflection point? If so, where is it, and what is the slope of the tangent at the inflection point?

(23) For which x with $0 \le x \le 3$ is
$$F(x) = \int_0^x t(3-t)(t-1)\,dt$$
maximal?

Integration – Areas, lengths, and volumes

(24) Find the length of the graph of $y = x^{5/4}$ with $0 \le x \le 16$.
(In this problem it doesn't say you can leave an integral in your answer, so you should compute the interal you get.)

(25) Find the length of the graph of $y = xe^x$ with $-1 \le x \le +1$ (you may leave an integral in your answer, after you simplify it as much as possible.)

(26) Find the total area of the bounded regions enclosed by the graphs $y = x$ and $y = x(1-x)(3-x)$.

(27) Find the total area of the bounded regions enclosed by the graphs $y = x$ and $y = x(1-x)(3+x)$.

(28) what is the volume of the solid obtained by rotating the region
$$\mathcal{R} = \{(x,y) \mid 1 \le x \le 3, -1 \le y \le (x-1)(x-2)(x-3)\}$$
around the y-axis.