MATH 221
FIRST SEMESTER
CALCULUS

fall 2007

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Math 221 – 1st Semester Calculus
Lecture notes version 1.0 (Fall 2007)

This is a self contained set of lecture notes for Math 221. The notes were written by Sigurd Angenent, starting from an extensive collection of notes and problems compiled by Joel Robbin.

The LaTeX and `Python` files which were used to produce these notes are available at the following web site

`www.math.wisc.edu/~angenent/Free-Lecture-Notes`

They are meant to be freely available in the sense that "free software" is free. More precisely:

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I. Numbers, Points, Lines and Curves

1. What is a number?

The basic objects that we deal with in calculus are the so-called “real numbers” which you have already seen in pre-calculus. To refresh your memory let’s look at the various kinds of “real” numbers that one runs into.

The simplest numbers are the ***positive integers***

$$1, 2, 3, 4, \cdots$$

the number ***zero***

$$0,$$

and the ***negative integers***

$$\cdots, -4, -3, -2, -1.$$

Together these form the integers or “whole numbers.”

Next, there are the numbers you get by dividing one whole number by another (nonzero) whole number. These are the so called fractions or ***rational numbers*** such as

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \ldots$$

or

$$-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}, -\frac{2}{4}, -\frac{3}{4}, -\frac{4}{3}, \ldots$$

By definition, any whole number is a rational number (in particular zero is a rational number.)

You can add, subtract, multiply and divide any pair of rational numbers and the result will again be a rational number (provided you don’t try to divide by zero).

One day in middle school you were told that there are other numbers besides the rational numbers, and the first example of such a number is the square root of two. It has been known ever since the time of the greeks that no rational number exists whose square is exactly 2, i.e. you can’t find a fraction $\frac{m}{n}$ such that

$$\left(\frac{m}{n}\right)^2 = 2, \text{ i.e. } m^2 = 2n^2.$$

Nevertheless, since

$$(1.4)^2 = 1.96 \text{ is less than 2, and}$$
$$(1.5)^2 = 2.25 \text{ is more than 2,}$$

it seems that there should be some number x between 1.4 and 1.5 whose square is exactly 2. So, we assume that there is such a number, and we call it the square root of 2, written as $\sqrt{2}$. This raises several questions. How do we know there really is a number between 1.4 and 1.5 for which $x^2 = 2$? How many other such numbers we are going to assume into existence? Do these new numbers obey the same algebra rules as the rational numbers? (e.g. when you add three numbers a, b and c the sum does not depend on the order in which you add them.) If we knew precisely what these numbers (like $\sqrt{2}$) were then we could perhaps answer such questions. It turns out to be rather difficult to give a precise description of what a number is, and in this course we won’t try to get anywhere near the bottom of this issue. Instead we will think of numbers as “infinite decimal expansions” as follows.

One can represent certain fractions as decimal fractions, e.g.

$$\frac{279}{25} = \frac{1116}{100} = 11.16.$$

Not all fractions can be represented as decimal fractions. For instance, expanding $\frac{1}{3}$ into a decimal fraction leads to an unending decimal fraction

$$\frac{1}{3} = 0.333\,333\,333\,333\,333\,\cdots$$

It is impossible to write the complete decimal expansion of $\frac{1}{3}$ because it contains infinitely many digits. But we can describe the expansion: each digit is a three. An electronic calculator, which always represents numbers as ***finite*** decimal numbers, can never hold the number $\frac{1}{3}$ exactly.

Every fraction can be written as a decimal fraction which may or may not be finite. If the decimal expansion doesn't end, then it must repeat. For instance,

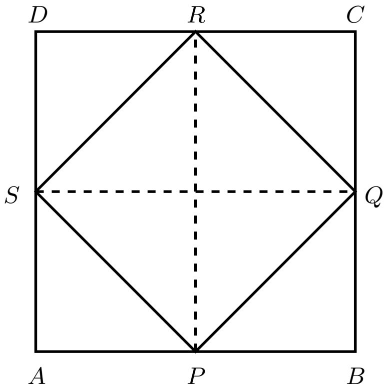
$$\frac{1}{7} = 0.142857\,142857\,142857\,142857\,\ldots$$

Conversely, any infinite repeating decimal expansion represents a rational number.

A ***real number*** is specified by a possibly unending decimal expansion. For instance,

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,9\ldots$$

Of course you can never write *all* the digits in the decimal expansion, so one only writes the first few digits and hides the others behind dots. To give a precise description of a real number (such as $\sqrt{2}$) you have to explain how one could in principle compute as many digits in the expansion as one would like. During the next three semesters of calculus we will not go into the details of how this should be done.



Another reason to believe in $\sqrt{2}$

The Pythagorean theorem says that the hypotenuse of a right triangle with sides 1 and 1 must be a line segment of length $\sqrt{2}$.

In middle or highschool you learned something similar to the following geometric construction of a line segment whose length is $\sqrt{2}$. Take a square $ABCD$ with sides of length 2. Let $PQRS$ be the square formed by connecting the midpoints of the square $ABCD$. Then the area of $PQRS$ is exactly half that of $ABCD$. Since $ABCD$ has area 4, the area of $PQRS$ must be 2, and therefore any side of $PQRS$ must have length $\sqrt{2}$.

Why are real numbers called real?

All the numbers we will use in this first semester of calculus are "real numbers." At some point (in 2nd semester calculus) it becomes useful to assume that there is a number whose square is -1. No real number has this property since the square of any real number is positive, so it was decided to call this new imagined number "imaginary" and to refer to the numbers we already have (rationals, $\sqrt{2}$-like things) as "real."

Exercises

1.1 – What is the 2007^{th} digit after the period in the expansion of $\frac{1}{7}$?

1.2 – Which of the following fractions have finite decimal expansions?

$$a = \frac{2}{3}, \quad b = \frac{3}{25}, \quad c = \frac{276937}{15625}.$$

1.3 – Write the numbers

$$x = 0.3131313131\ldots, y = 0.273273273273\ldots \text{ and } z = 0.2154154154154154154\ldots$$

as fractions (i.e. write them as $\frac{m}{n}$, specifying m and n.)

(Hint: show that $100x = x + 31$. A similar trick works for y, but z is a little harder.)

1.4 – Is the number whose decimal expansion after the period consists only of nines, i.e.

$$x = 0.99999999999999999\ldots$$

an integer?

1.5 – There is a real number x which satisfies

$$\tfrac{1}{3}x^7 + x + 2 = 5.$$

Find the first three digits in the decimal expansion of x. [Use a calculator.]

2. The real number line and intervals

It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the ***origin***. We also decide which direction we call "left" and hence which we call "right." Some draw the number line vertically and use the words "up" and "down."

To plot any real number x one marks off a distance x from the origin, to the right (up) if $x > 0$, to the left (down) if $x < 0$.

The ***distance along the number line*** between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number.

Almost every equation involving variables x, y, etc. we write down in this course will be true for some values of x but not for others. In modern abstract mathematics a collection of real numbers (or any other kind of mathematical objects) is called a ***set***. Below are some examples of sets of real numbers. We will use the notation from these examples throughout this course.

2.1. *Intervals*

The collection of all real numbers between two given real numbers form an interval. The following notation is used

- (a, b) is the set of all real numbers x which satisfy $a < x < b$.
- $[a, b)$ is the set of all real numbers x which satisfy $a \leq x < b$.
- $(a, b]$ is the set of all real numbers x which satisfy $a < x \leq b$.
- $[a, b)$ is the set of all real numbers x which satisfy $a \leq x \leq b$.

If the endpoint is not included then it may be ∞ or $-\infty$. E.g. $(-\infty, 2]$ is the interval of all real numbers (both positive and negative) which are ≤ 2.

2.2. *Set notation*

A common way of describing a set is to say it is the collection of all real numbers which satisfy a certain condition. One uses this notation

$$\mathcal{A} = \bigl\{x \mid x \text{ satisfies this or that condition}\bigr\}$$

Most of the time we will use upper case letters in a calligraphic font to denote sets. ($\mathcal{A}$,$\mathcal{B}$,$\mathcal{C}$,$\mathcal{D}$, ...)

For instance, the interval (a, b) can be described as

$$(a, b) = \bigl\{x \mid a < x < b\bigr\}$$

The set

$$\mathcal{B} = \bigl\{x \mid x^2 - 1 > 0\bigr\}$$

consists of all real numbers x for which $x^2 - 1 > 0$, i.e. it consists of all real numbers x for which either $x > 1$ or $x < -1$ holds. This set consists of two parts: the interval $(-\infty, -1)$ and the interval $(1, \infty)$.

You can try to draw a set of real numbers by drawing the number line and coloring the points belonging to that set red, or by marking them in some other way.

Some sets can be very difficult to draw. Consider

$$\mathcal{C} = \bigl\{x \mid x \text{ is a rational number}\bigr\}$$

or

$$\mathcal{D} = \bigl\{x \mid \text{the number 3 does not appear in the decimal expansion of } x\bigr\}.$$

Sets can also contain just a few numbers, like

$$\mathcal{E} = \{1, 2, 3\}$$

which is the set containing the numbers one, two and three. Or the set

$$\mathcal{F} = \bigl\{x \mid x^3 - 4x^2 + 1 = 0\bigr\}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are two sets then ***the union of*** $\mathcal{A}$ ***and*** $\mathcal{B}$ is the set which contains all numbers that belong either to $\mathcal{A}$ or to $\mathcal{B}$. The following notation is used

$$\mathcal{A} \cup \mathcal{B} = \bigl\{x \mid x \text{ belongs to } \mathcal{A} \text{ or to } \mathcal{B}.\bigr\}$$

Similarly, the ***intersection of two sets*** $\mathcal{A}$ ***and*** $\mathcal{B}$ is the set of numbers which belong to both sets. This notation is used:

$$\mathcal{A} \cup \mathcal{B} = \bigl\{x \mid x \text{ belongs to both } \mathcal{A} \text{ and } \mathcal{B}.\bigr\}$$

Exercises

2.1 – Draw the following sets of real numbers

$$\mathcal{A} = \{x \mid x^2 - 3x + 2 \le 0\} \qquad \mathcal{B} = \{x \mid x^2 - 3x + 2 \ge 0\}$$
$$\mathcal{C} = \{x \mid x^2 - 3x > 3\} \qquad \mathcal{D} = \{x \mid x^2 - 5 > 2x\}$$
$$\mathcal{E} = \{t \mid t^2 - 3t + 2 \le 0\} \qquad \mathcal{F} = \{\alpha \mid \alpha^2 - 3\alpha + 2 \ge 0\}$$
$$\mathcal{G} = (0,1) \cup (5,7] \qquad \mathcal{H} = \big(\{1\} \cup \{2,3\}\big) \cap (0, 2\sqrt{2})$$
$$\mathcal{P} = \{x^2 - 2x \mid 0 \le x \le 2\} \qquad \mathcal{Q} = \{x^2 - 2x \mid 0 \le x \le 1\}$$
$$\mathcal{R} = \{\theta \mid \sin\theta = \tfrac{1}{2}\} \qquad \mathcal{S} = \{\varphi \mid \cos\varphi > 0\}$$

2.2 – Suppose $\mathcal{A}$ and $\mathcal{B}$ are intervals. Is it always true that $\mathcal{A} \cap \mathcal{B}$ is an interval? How about $\mathcal{A} \cup \mathcal{B}$?

2.3 – Consider the sets

$$\mathcal{M} = \{x \mid x > 0\} \text{ and } \mathcal{N} = \{y \mid y > 0\}.$$

Are these sets the same?

3. Sets of Points in the Plane

3.1. *Cartesian Coordinates*

The coordinate plane with its x and y axes are familiar from middle/high school mathematics.

Briefly, you can specify the location of any point in the plane by choosing two fixed orthogonal lines (called the x and y axes) and specifying the distances to each of the two axes. The distance to the x axes is y, and the distance to the y axis is x. By allowing the numbers x and y to be either positive or negative, one can keep track on which side of the axes the point with coordinates x and y lies.

The notation $P(x, y)$ is used as an abbreviation for the more cumbersome phrase "the point P whose coordinates are (x, y)."

3.2. *Sets*

Just as one can consider sets of real numbers, one can also consider sets of points in the plane. Examples of such sets are

$\mathcal{A} =$ All points on the x-axis

$\mathcal{B} =$ All points on the x or y-axes

$\mathcal{C} =$ All points whose distance to the origin is 25

$\mathcal{D} =$ All points whose coordinates x and y are both integers

$\mathcal{E} =$ All points one of whose coordinates x and y is an integer

$\mathcal{F} =$ All points whose coordinates x and y are both rational numbers

One can also write some of these sets as follows

$$\mathcal{A} = \{(x, y) \mid y = 0\} = \{(x, 0) \mid x \text{ arbitrary}\}$$
$$\mathcal{B} = \{(x, y) \mid x = 0 \text{ or } y = 0\} = \{(x, y) \mid xy = 0\}$$
$$\mathcal{C} = \Big\{(x, y) \mid \sqrt{x^2 + y^2} = 25\Big\} = \big\{(x, y) \mid x^2 + y^2 = 625\big\}.$$

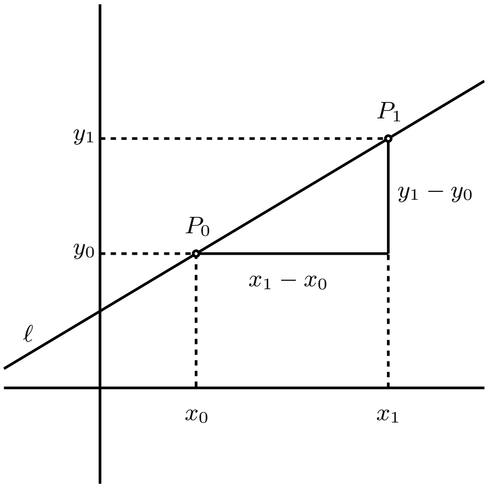


Figure 1. A straight line and its slope

3.3. *Lines*

A straight line through two points is also an example of a set of points in the plane. Unless the line is "vertical," (i.e. parallel to the y-axis) it is a set of the form

$$\ell = \{(x, y) \mid y = mx + n\} \tag{1}$$

where m is the ***slope*** of the line, and n is its y-intercept (the y coordinate of the point where ℓ intersects the $;y$ axis).

If $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ are two points on the line ℓ, then one can compute the slope m from the "rise-over-run" formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

This formula actually contains a theorem, namely it says that the ratio $(y_1 - y_0) : (x_1 - x_0)$ is the same for every pair of points (x_0, y_0) and (x_1, y_1) that you could pick on the line.

Exercises

3.1 – Draw the sets from §3.2.

3.2 – What is the distance between the points $P(1, 3)$ and $Q(-3, 7)$?

Find the equation for the line ℓ through P and Q.

3.3 – Let ℓ be the line through $A(0, 1)$ and $B(2, 3)$, and let ℓ' be the line through $C(-2, -10)$ and $D(3, 3)$.

Find the equations for ℓ and ℓ'.

Where do the two lines ℓ and ℓ' intersect?

Draw the sets $\mathcal{A} = \ell \cup \ell'$ and $\mathcal{B} = \ell \cap \ell'$.

3.4 – Let ℓ be the line through the point $(-1, 0)$ with slope m, and let $\mathcal{C}$ be the circle centered at the origin with radius 1.

Make a drawing of ℓ and C, and find $\ell \cap \mathcal{C}$.

3.5 – Draw the following sets of points in the plane

$$\mathcal{A} = \{(x,y) \mid y > 0 \text{ and } x > 0\} \qquad \mathcal{B} = \{(x,y) \mid y < 0 \text{ or } y > x\}$$
$$\mathcal{C} = \{(x,y) \mid y + x = 0\} \qquad \mathcal{D} = \{(x,y) \mid x^2 - 5 > 2x\}$$
$$\mathcal{E} = \{(x,y) \mid x^2 + y^2 < 1\} \qquad \mathcal{F} = \{(x,y) \mid (x-1)^2 + y^2 = 1\}$$
$$\mathcal{G} = \mathcal{E} \cup \mathcal{F} \qquad \mathcal{H} = \mathcal{E} \cap \mathcal{F}$$

4. Functions

Definition. 4.1. *A **function** consists of a **rule** and a **domain**.*

*The **domain** is a set of real numbers, and the **rule** tells you how to compute a real number $f(x)$ for any given real number x in the domain.*

*The set of all possible numbers $f(x)$ as x runs over the domain is called the **range** of the function.*

The rule which specifies a function can come in many different forms. Most often it is a formula, as in

$$f(x) = x^2 - 2x + 1, \quad \text{domain of } f = \text{all real numbers.}$$

or a few formulas, as in

$$g(x) = \begin{cases} 2x & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases} \qquad \text{domain of } g = \text{all real numbers.}$$

Functions which are defined by different formulas on different intervals are sometimes called "piecewise defined functions."

In this course we will usually not be careful about specifying the domain of the function. When this happens the domain is understood to be the set of all x for which the rule which tells you how to compute $f(x)$ is meaningful. For instance, if we say that h is the function

$$h(x) = \sqrt{x}$$

then the domain of h is understood to be the set of all nonnegative real numbers

$$\text{domain of } h = [0, \infty)$$

since $\sqrt{x}$ is well-defined for all $x \ge 0$ and undefined for $x < 0$.

A systematic way of finding the domain and range of a function for which you are only given a formula is as follows:

- The domain of f consists of all x for which $f(x)$ is well-defined ("makes sense")
- The range of f consists of all y for which you can solve the equation $f(x) = y$.

4.1. *Example: Find the domain and range of $f(x) = 1/x^2$*

The expression $1/x^2$ can be computed for all real numbers x except $x = 0$ since this leads to division by zero. Hence the domain of the function $f(x) = 1/x^2$ is

$$\{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty).$$

To find the range we ask "for which y can we solve the equation $y = f(x)$ for x," i.e. we try to solve

$$y = \frac{1}{x^2}$$

for x.

Since x cannot be zero we have $x^2 > 0$ for all values of x we are allowed to choose. Therefore $1/x^2 > 0$, and the equation $y = 1/x^2$ will not have a solution if $y \le 0$. On the other hand, if $y > 0$ then the equation has not just one but two solutions, namely $x = \pm 1/\sqrt{y}$. So any positive number y belongs to the range of f, and if y is not positive, then y does not belong to the range of f. We have found that

$$\mathrm{range}(f) = (0, \infty).$$

4.2. *Functions in "real life"*

One can describe the motion of an object using a function. If some object is moving along a straight line, then you can define the following function: Let $x(t)$ be the distance from the object to a fixed marker on the line, at the time t. Here the domain of the function is the set of all times t for which we know the position of the object, and the rule is

Given t, measure the distance between the object and the marker.

There are many examples of this kind. For instance, a biologist could describe the growth of a cell by defining $m(t)$ to be the mass of the cell at time t (measured since the birth of the cell). Here the domain is the interval $[0, T]$, where T is the life time of the cell, and the rule that describes the function is *given t, weigh the cell at time t.*

5. **The graph of a function**

Definition. 5.1. *The **graph** of a function*

$$y = f(x)$$

is the set of all points $P(x, y)$ whose coordinates (x, y) satisfy the equation $y = f(x)$, i.e.

$$\mathrm{Graph}(f) = \big\{(x, y) \mid y = f(x)\big\}.$$

5.1. *Vertical Line Property*

Generally speaking graphs of functions are curves in the plane but they distinguish themselves from arbitrary curves (or point sets) by the way they intersect vertical lines: *The graph of a function cannot intersect a vertical line* $x =$ constant *in more than one point.* To see why this is so, suppose that you have two points (x_0, y_0) and (x_1, y_1) on the graph of f which also lie on the same vertical line with equation $x = a$.

Since both points lie on the vertical line with $x = a$ we have

$$x_0 = a \text{ and } x_1 = a$$

Since both points lie on the graph of f we also have

$$y_0 = f(x_0) \text{ and } y_1 = f(x_1).$$

It follows that $x_0 = x_1 = a$ and $y_0 = f(x_0) = f(a)$ and $y_1 = f(x_1) = f(a)$ so that both points are given by $(x_0, y_0) = (x_1, y_1) = (a, f(a))$, i.e. they are the same point.

5.2. *Example*

The point set determined by the equation $x^2 + y^2 = 1$ is a circle; it is not the graph of a function since the vertical line $x = 0$ (the y-axis) intersects the graph in two points $P_1(0, 1)$ and $P_2(0, -1)$.

The graph of $f(x) = x^3 - x$ "goes up and down," and while it intersects the x-axis in three points ($(-1, 0)$, $(0, 0)$ and $(1, 0)$) it intersects every vertical line in exactly one point.

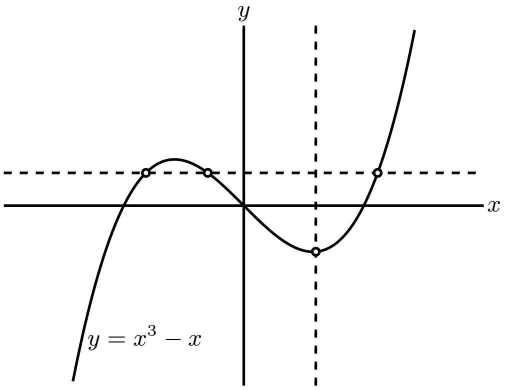


FIGURE 2. The graph of a function intersects vertical lines in at most one point, but may intersect horizontal lines more than once

6. **Inverse functions and Implicit functions**

For many functions the rule which tells you how to compute it is not an explicit formula, but instead an equation which you still must solve. A function which is defined in this way is called an "implicit function."

6.1. *Example*

One can define a function f by saying that for each x the value of $f(x)$ is the solution y of the equation

$$x^2 + 2y - 3 = 0.$$

In this example you can solve the equation for y,

$$y = \frac{3 - x^2}{2}.$$

Thus we see that the function we have defined is $f(x) = (3 - x^2)/2$.

Here we have two definitions of the same function, namely (i) "f is defined by $y = f(x) \iff x^2 + 2y - 3 = 0$" and (ii) "$f$ is defined by $f(x) = (3 - x^2)/2$." The first definition is the implicit definition, the second is explicit. You see that with an "implicit function" it isn't the function itself, but rather the way it was defined that's implicit.

6.2. *Another example: domain of an implicitly defined function*

Define g by saying that for any x the value $y = g(x)$ is the solution of

$$x^2 + xy - 3 = 0.$$

Just as in the previous example one can then solve for y, and one finds that

$$g(x) = y = \frac{3 - x^2}{x}.$$

Unlike the previous example this formula does not make sense when $x = 0$, and indeed, for $x = 0$ our rule for g says that $g(0) = y$ is the solution of

$$0^2 + 0 \cdot y - 3 = 0, \text{ i.e. } y \text{ is the solution of } 3 = 0.$$

That equation has no solution and hence $x = 0$ does not belong to the domain of our function g.

FIGURE 3. The circle determined by $x^2 + y^2 = 1$ is not the graph of a function, but it is the union of the graphs of the two functions $h_1(x) = \sqrt{1-x^2}$ and $h_2(x) = -\sqrt{1-x^2}$.

6.3. *Example: the equation alone does not determine the function*

Define $y = h(x)$ to be the solution of

$$x^2 + y^2 = 1.$$

If $x > 1$ or $x < -1$ then $x^2 > 1$ and there is no solution, so $h(x)$ is at most defined when $-1 \leq x \leq 1$. But when $-1 < x < 1$ there is another problem: not only does the equation have a solution, but it even has two solutions:

$$x^2 + y^2 = 1 \iff y = \sqrt{1-x^2} \text{ or } y = -\sqrt{1-x^2}.$$

The rule which defines a function must be unambiguous, and since we have not specified which of these two solutions is $h(x)$ the function is not defined for $-1 < x < 1$.

One can fix this by making a choice, but there are many possible choices. Here are three possibilities:

$$\begin{aligned} h_1(x) &= \text{the nonnegative solution } y \text{ of } x^2 + y^2 = 1 \\ h_2(x) &= \text{the nonpositive solution } y \text{ of } x^2 + y^2 = 1 \\ h_3(x) &= \begin{cases} h_1(x) & \text{when } x < 0 \\ h_2(x) & \text{when } x \geq 0 \end{cases} \end{aligned}$$

6.4. *Why use implicit functions?*

In all the examples we have done so far we could replace the implicit description of the function with an explicit formula. This is not always possible or if it is possible the implicit description is much simpler than the explicit formula. For instance, you can define a function f by saying that $y = f(x)$ if and only if

$$y^3 + 3y + 2x = 0. \tag{2}$$

This means that the recipe for computing $f(x)$ for any given x is "solve the equation $y^3 + 3y + 2x = 0$." E.g. to compute $f(0)$ you set $x = 0$ and solve $y^3 + 3y = 0$. The only solution is $y = 0$, so $f(0) = 0$. To compute $f(1)$ you have to solve $y^3 + 3y + 2 \cdot 1 = 0$, and if you're lucky you see that $y = -1$ is the solution, and $f(1) = -1$.

In general, no matter what x is, the equation (2) turns out to have exactly one solution y (which depends on x, this is how you get the function f). Solving (2) is not easy. In the early 1500s Cardano and Tartaglia discovered a formula[1] for the solution. Here it is:

$$y = f(x) = \sqrt[3]{-x + \sqrt{1+x^2}} - \sqrt[3]{x + \sqrt{1+x^2}}.$$

The implicit description looks a lot simpler, and when we try to differentiate this function later on, it will be much easier to use "implicit differentiation" than to use the Cardano-Tartaglia formula directly.

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[1] To see the solution and its history visit
`www.gap-system.org/~history/HistTopics/Quadratic_etc_equations.html`

6.5. *Inverse functions*

If you have a function f, then you can try to define a new function g which will be called the ***inverse function of*** f by requiring

$$y = g(x) \iff x = f(y).$$

In other words, to find $y = g(x)$ you solve the equation $x = f(y)$. Depending on the function f it may happen that the equation $x = f(y)$ has no solution, or that it has more than one solution. In either case we won't get an unambiguous value for y. However, if it is the case for some x that the equation $x = f(y)$ has ***exactly one solution*** y, then we can define $g(x) = y$. The inverse function of f is usually written as f^{-1}.

More precisely:

Definition. 6.1. *If f is a given function then the domain of the inverse function of f consists of every number x for which the equation*

$$x = f(y) \tag{3}$$

has exactly one solution.

If (3) has exactly one solution y then $f^{-1}(x) = y$.

6.6. *Examples*

Consider the function f with $f(x) = 2x + 3$. Then the equation $f(y) = x$ works out to be

$$2y + 3 = x$$

and this has the solution

$$y = \frac{x-3}{2}.$$

So $f^{-1}(x)$ is defined for all x, and it is given by $f^{-1}(x) = (x-3)/2$.

Next consider the function $g(x) = x^2$ with domain all real numbers. To see if this function has an inverse we try to solve the equation $g(y) = x$, i.e. we try to solve $y^2 = x$. If $x > 0$ then this equation has ***two*** solutions, $\pm\sqrt{x}$; if $x < 0$ then $y^2 = x$ has no solutions; if $x = 0$ then $y^2 = x$ has exactly one solution, namely $y = 0$. So we see that $g^{-1}(x)$ is only defined when $x = 0$. For all other x the equation defining $g^{-1}(x)$ either gives too few or too many solutions.

We now consider the function g again ***but we change its domain,*** i.e. we consider the function h defined by $h(x) = x^2$, and whose domain is $[0, \infty)$. So h is defined by the same rule as g ("square whatever number you are given"), but h is only allowed to apply the rule to nonnegative numbers, while g was allowed to apply its rule to all numbers.

What is the inverse of h? To find $h^{-1}(x)$ we solve the equation $h(y) = x$, i.e. $y^2 = x$. For negative x we again get no solution, just as in the case of g. But when x is positive we now get something else: the equation $h(y) = x$ ***has only one solution***. There are two numbers y which satisfy $y^2 = x$, but only one of them lies in the domain of h.

Conclusion: the inverse of h has domain $[0, \infty)$ and is given by $h(x) = \sqrt{x}$.

6.7. *Inverse trigonometric functions*

The familiar trigonometric functions Sine, Cosine and Tangent have inverses which are called arcsine, arccosine and arctangent.

$$\begin{array}{llll}
y = f(x) & & x = f^{-1}(y) & \\
y = \sin x & (-\pi/2 \le x \le \pi/2) & x = \arcsin(y) & (-1 \le y \le 1) \\
y = \cos x & (0 \le x \le \pi) & x = \arccos(y) & (-1 \le y \le 1) \\
y = \tan x & (-\pi/2 < x < \pi/2) & x = \arctan(y) &
\end{array}$$

The notations $\arcsin y = \sin^{-1} y$, $\arccos x = \cos^{-1} x$, and $\arctan u = \tan^{-1} u$ are also commonly used for the inverse trigonometric functions. We will avoid the $\sin^{-1} y$ notation because it is ambiguous. Namely, everybody writes the square of $\sin y$ as

$$\bigl(\sin y\bigr)^2 = \sin^2 y.$$

This suggests that $\sin^{-1} y$ should mean

$$\sin^{-1} y = \bigl(\sin y\bigr)^{-1} = \frac{1}{\sin y},$$

and not $\arcsin y$.

Exercises

6.1 – Draw the graphs of the functions h_1, h_2, h_3 from §6.3

6.2 – Find a formula for the function f which is defined by

$$y = f(x) \iff x^2y + y = 7.$$

What is the domain of f?

6.3 – Find a formula for the function f which is defined by

$$y = f(x) \iff x^2y - y = 6.$$

What is the domain of f?

6.4 – Let f be the function defined by $y = f(x) \iff y$ is the largest solution of

$$x^2 + xy + y^2 = 0.$$

Find a formula for f. What are the domain and range of f?

6.5 – Find a formula for the function f which is defined by

$$y = f(x) \iff 2x + 2xy + y^2 = 5 \text{ and } y > -x.$$

Find the domain of f.

6.6 – Use a calculator to compute $f(1.2)$ in three decimals where f is the implicitly defined function from §6.4. (There are (at least) two different ways of finding $f(1.2)$)

<u>*6.7*</u> – On a graphing calculator plot the graphs of the following functions, and explain the results.

$$\begin{aligned}
f(x) &= \arcsin(\sin x) && -2\pi \le x \le 2\pi \\
g(x) &= \arcsin(x) + \arccos(x) && 0 \le x \le 1 \\
h(x) &= \arctan \frac{\sin x}{\cos x} && |x| < \pi/2 \\
k(x) &= \arctan \frac{\cos x}{\sin x} && |x| < \pi/2 \\
l(x) &= \arcsin(\cos x) && -\pi \le x \le \pi \\
m(x) &= \cos(\arcsin x) && -1 \le x \le 1
\end{aligned}$$

<u>*6.8*</u> – A function f is given which satisfies

$$f(2x+3) = x^2$$

for all real numbers x.

Compute

(a) $f(0)$	(b) $f(3)$	(c) $f(x)$
(d) $f(y)$	(e) $f(f(2))$	

where x and y are arbitrary real numbers.

What are the range and domain of f?

<u>*6.9*</u> – A function f is given which satisfies

$$f\left(\frac{1}{x+1}\right) = 2x - 12.$$

for all real numbers x.

Compute

(a) $f(1)$	(b) $f(0)$	(c) $f(x)$
(d) $f(t)$	(e) $f(f(2))$	

where x and t are arbitrary real numbers.

What are the range and domain of f?

<u>*6.10*</u> – Does there exist a function f which satisfies

$$f(x^2) = x + 1$$

for ***all*** real numbers x?

II. Derivatives (1)

To work with derivatives you have to know what a limit is, but to motivate why we are going to study limits let's first look at the two classical problems that gave rise to the notion of a derivative: the tangent to a curve, and the instantaneous velocity of a moving object.

7. The tangent to a curve

Suppose you have a function $y = f(x)$ and you draw its graph. If you want to find the tangent to the graph of f at some given point on the graph of f, how would you do that?

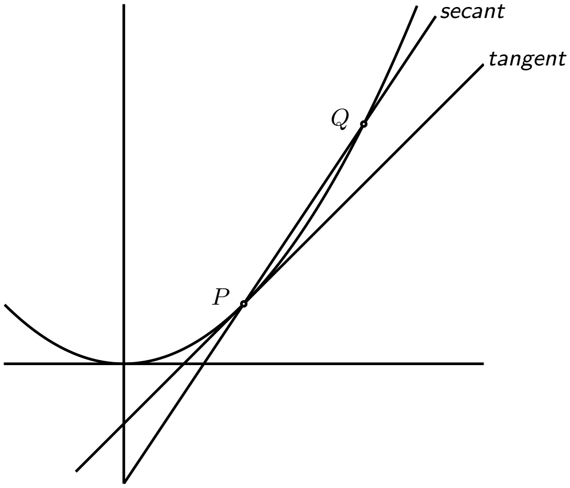


FIGURE 4. Constructing the tangent by letting $Q \to P$

Let P be the point on the graph at which want to draw the tangent. If you are making a real paper and ink drawing you would take a ruler, make sure it goes through P and then turn it until it doesn't cross the graph anywhere else.

If you are using equations to describe the curve and lines, then you could pick a point Q on the graph and construct the line through P and Q ("construct" means "find an equation for"). This line is called a "secant," and it is of course not the tangent that you're looking for. But if you choose Q to be very close to P then the secant will be close to the tangent.

So this is our recipe for constructing the tangent through P: pick another point Q on the graph, find the line through P and Q, and see what happens to this line as you take Q closer and closer to P. The resulting secants will then get closer and closer to some line, and that line is the tangent.

We'll write this in formulas in a moment, but first let's worry about how close Q should be to P. We can't set Q equal to P, because then P and Q don't determine a line (you need ***two*** points to determine a line). If you choose Q different from P then you don't get the tangent, but at best something that is "close" to it. Some people have suggested that one should take Q "infinitely close" to P, but it isn't clear what that would mean. The concept of a limit is meant to solve this confusing problem.

8. An example – tangent to a parabola

To make things more concrete, suppose that the function we had was $f(x) = x^2$, and that the point was $(1, 1)$. The graph of f is of course a parabola.

Any line through the point $P(1, 1)$ has equation

$$y - 1 = m(x - 1)$$

where m is the slope of the line. So instead of finding the equation of the secant and tangent lines we will find their slopes.

Let Q be the other point on the parabola, with coordinates (x, x^2). We can "move Q around on the graph" by changing x. Whatever x we choose, it must be different from 1, for otherwise P and Q would be the same point. What we want to find out is how the line through P and Q changes if x is changed (and in particular, if x is chosen very close to a). Now, as one changes x one thing stays the same, namely, the secant still goes through P. So to describe the secant we only need to know its slope. By the "rise over run" formula, the slope of the secant line joining P and Q is

$$m_{PQ} = \frac{\Delta y}{\Delta x} \tag{4}$$

where

$$\Delta y = x^2 - 1$$

is the difference between the vertical coordinate x^2 of Q and the vertical coordinate 1 of P and

$$\Delta x = x - 1$$

is the difference of the horizontal coordinates of P and Q. By factoring $x^2 - 1$ we can rewrite the formula for the slope as follows

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1. \tag{5}$$

As x gets closer to 1, the slope m_{PQ}, being $x + 1$, gets closer to the value $1 + 1 = 2$. We say that

the limit of the slope m_{PQ} ***as*** Q ***approaches*** P ***is*** 2.

In symbols,

$$\lim_{Q \to P} m_{PQ} = 2,$$

or, since Q approaching P is the same as x approaching 1,

$$\lim_{x \to 1} m_{PQ} = 2. \tag{6}$$

So we find that the tangent line to the parabola $y = x^2$ at the point $(1, 1)$ has equation

$$y - 1 = 2(x - 1), \text{ i.e. } y = 2x - 1.$$

8.1 – Repeat the above reasoning to find the slope at the point $(\frac{1}{2}, \frac{1}{4})$, or more generally at any point (a, a^2) on the parabola.

A warning: you cannot substitute $x = 1$ in equation (5) to get (6) even though it looks like that's what we did. The reason why you can't do that is that when $x = 1$ the point Q coincides with the point P so "the line through P and Q" is not defined; also, if $x = 1$ then $\Delta x = \Delta y = 0$ so that the rise-over-run formula for the slope gives

$$m_{PQ} = \frac{\Delta x}{\Delta y} = \frac{0}{0} = \text{ undefined.}$$

It is only after the algebra trick in (5) that setting $x = 1$ gives something that is well defined. But if the intermediate steps leading to $m_{PQ} = x + 1$ aren't valid for $x = 1$ why should the final result be worth anything for $x = 1$?

Something more complicated has happened. We did a calculation which is valid for all $x \neq 1$, and later looked at what happens if x gets "very close to 1." This is the concept of a limit and we'll study it in more detail later in this section, but first another example.

9. Instantaneous velocity

If you try to define "instantaneous velocity" you will again end up trying to divide zero by zero. Here is how it goes: When you are driving in your car the speedometer tells you how fast your are going, i.e. what your velocity is. What is this velocity? What does it mean if the speedometer says "50mph"?

We all know what ***average velocity*** is. Namely, if it takes you two hours to cover 100 miles, then your average velocity was

$$\frac{\text{distance traveled}}{\text{time it took}} = 50 \text{ miles per hour.}$$

This is not the number the speedometer provides you – it doesn't wait two hours, measure how far you went and compute distance/time. If the speedometer in your car tells you that you are driving 50mph, then that should be your velocity ***at the moment*** that you look at your speedometer, i.e. "distance traveled over time it took" at the moment you look at the speedometer. But during the moment you look at your speedometer no time goes by (because a moment has no length) and you didn't cover any distance, so your velocity at that moment is $\frac{0}{0}$, i.e. undefined. Your velocity at ***any*** moment is undefined. But then what is the speedometer telling you?

To put all this into formulas we need to introduce some notation. Let t be the time (in hours) that has passed since we got onto the road, and let $s(t)$ be the distance we have covered since then.

Instead of trying to find the velocity exactly at time t, we find a formula for the average velocity during some (short) time interval beginning at time t. We'll write Δt for the length of the time interval.

At time t we have traveled $s(t)$ miles. A little later, at time $t + \Delta t$ we have traveled $s(t+\Delta t)$. Therefore during the time interval from t to $t+\Delta t$ we have moved $s(t+\Delta t)-s(t)$ miles. Our average velocity in that time interval is therefore

$$\frac{s(t + \Delta t) - s(t)}{\Delta t} \text{ miles per hour.}$$

The shorter you make the time interval, i.e. the smaller you choose Δt, the closer this number should be to the instantaneous velocity at time t.

So we have the following formula (definition, really) for the velocity at time t

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}. \tag{7}$$

10. Rates of change

The two previous examples have much in common. If we ignore all the details about geometry, graphs, highways and motion, the following happened in both examples:

We had a function $y = f(x)$, and we wanted to know how much $f(x)$ changes if x changes. If you change x to $x + \Delta x$, then y will change from $f(x)$ to $f(x + \Delta x)$. The change in y is therefore

$$\Delta y = f(x + \Delta x) - f(x),$$

and the average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{8}$$

This is the average rate of change of f over the interval from x to $x + \Delta x$. To define ***the rate of change of the function f at x*** we let the length Δx of the interval become smaller and smaller, in the hope that the average rate of change over the shorter and shorter time intervals will get closer and closer to some number. If that happens then that "limiting number" is called the rate of change of f at x, or, the ***derivative*** of f at x. It is written as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{9}$$

Derivatives and what you can do with them are what the first half of this semester is about. The description we just went through shows that to understand what a derivative is you need to know what a limit is. In this chapter we'll study limits so that we get a less vague understanding of formulas like (9).

Exercises

10.1 – Using a calculator (if you must) approximate the derivatives of the following functions at the indicated points (i.e. find $f'(a)$) by computing $\frac{\Delta y}{\Delta x}$ for various values of Δx.

(a) $f(x) = x^2 - 2x + 1,\ \ a = 1$ $\qquad$ (b) $f(x) = x^2,\ \ a = -1$

(a) $f(x) = x^{33},\ \ a = 1$ $\qquad$ (d) $f(x) = 2^x,\ \ a = 1$

Use the following values Δx

$$\Delta x = 0.1, 0.01, 0.001, 10^{-6}, 10^{-12}.$$

10.2 – Simplify the algebraic expressions you get when you compute Δy and $\Delta y/\Delta x$ for the following functions

(a) $y = x^2 - 2x + 1$ $\qquad$ (b) $y = \sin x$

(c) $y = \dfrac{1}{x}$ $\qquad$ (d) $y = 2^x$

10.3 – Suppose that some quantity y is a function of some other quantity x, and suppose that y is a mass, i.e. y is measured in pounds, and x is a length, measured in feet. What units do the increments Δy and Δx, and the derivative dy/dx have?

10.4 – Let $A(r)$ be the area enclosed by a circle of radius r, and let $L(r)$ be the length of the circle. Show that

$$A'(r) = L(r).$$

III. Limits and Continuous Functions

11. Informal definition of limits

While it is easy to define precisely in a few words what a square root is ($\sqrt{a}$ is the positive number whose square is a) the definition of the limit of a function runs over several terse lines, and most people don't find it very enlightening when they first see it. (See §12.) So we postpone this for a while and fine tune our intuition for another page.

Definition. 11.1 (Definition of limit (1st attempt)). *If g is some function then*

$$\lim_{x\to a} g(x) = L$$

is read "the limit of $g(x)$ as x approaches a is L." It means that if you choose values of x which are close ***but not equal*** *to a, then $g(x)$ will be close to the value L; moreover, $g(x)$ gets closer and closer to L as x gets closer and closer to a.*

The following alternative notation is sometimes used

$$g(x) \to L \quad \text{as} \quad x \to a;$$

(read "$g(x)$ approaches L as x approaches a" or "$g(x)$ goes to L as x goes to a".)

11.1. *Example*

If $g(x) = x + 3$ then

$$\lim_{x\to 4} g(x) = 7,$$

is true, because if you substitute number x close to 4 in $g(x) = x + 3$ the result will be close to 7.

11.2. *Example: substituting numbers to guess a limit*

What (if anything) is

$$\lim_{x\to 2} \frac{x^2 - 2x}{x^2 - 4}?$$

Here $g(x) = (x^2 - 2x)/(x^2 - 4)$ and $a = 2$.

We first try to substitute $x = 2$, but this leads to

$$g(2) = \frac{2^2 - 2\cdot 2}{2^2 - 4} = \frac{0}{0}$$

which does not exist. Next we try to substitute values of x close but not equal to 2. Table 1 suggests that $g(x)$ approaches 0.5.

x	$g(x)$	x	$h(x)$
3.000000	0.600000	1.000000	1.009990
2.500000	0.555556	0.500000	1.009980
2.100000	0.512195	0.100000	1.009899
2.010000	0.501247	0.010000	1.008991
2.001000	0.500125	0.001000	1.000000

TABLE 1. Finding limits by substituting values of x "close to a"

11.3. *Example: Substituting numbers can suggest the wrong answer*

The previous example shows that our first definition of "limit" is not very precise, because it says "x close to a," but how close is close enough? Suppose we had taken the function

$$h(x) = \frac{101\,000x}{100\,000x + 1}$$

and we had asked for the limit $\lim_{x\to 0} h(x)$.

Then substitution of some "small values of x" could lead us to believe that the limit is $1.000\ldots$. Only when you substitute even smaller values do you find that the limit is 0 (zero)!

Exercise

11.1 – Our definition of a derivative in (9) contains a limit. What is the function g there, and what is the variable?

Answer: Δx is the variable, and $g(\Delta x) = (f(x + \Delta x) - f(x))/\Delta x$; we want $\lim_{\Delta x\to 0} g(\Delta x)$.

12. **The formal, authoritative, definition of limit**

The informal description of the limit uses phrases like "closer and closer" and "really very small." In the end we don't really know what they mean, although they are suggestive. "Fortunately" there is a good definition, i.e. one which is unambiguous and can be used to settle any dispute about the question of whether $\lim_{x\to a} g(x)$ equals some number L or not. Here is the definition. It takes a while to digest, so read it once, look at the examples, do a few exercises, read the definition again. Go on to the next sections. Throughout the semester come back to this section and read it again.

Definition. 12.1. *Suppose we have an interval $p < x < q$ and a number a in that interval $p < a < q$. Let g be a function which is defined for all x in the interval $p < x < q$, except possibly at $x = a$.*

We say that L is the limit of $g(x)$ as $x \to a$, if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that for all x in the interval $p < x < q$ one has

$$|x - a| < \delta \;\&\; x \neq a \implies |g(x) - L| < \varepsilon.$$

Why the absolute values? The quantity $|x-y|$ is the distance between the points x and y on the number line, and one can measure how close x is to y by calculating $|x - y|$.

What are ε and δ? The quantity ε is how close you would like $g(x)$ to be to its limit L; the quantity δ is how close you have to choose x to a to achieve this. If you make ε smaller then you are requiring $g(x)$ to be closer to L than before, and hence you may have to restrict the allowed values of x, and thus reduce δ. So, δ ***depends on*** ε. See also figures 5 and 6.

A strategy for proving $\lim_{x\to a} g(x) = L$. using what you know about the function g write out the expression $|g(x) - L|$, and try to simplify it so that you get something of the form

$$|g(x) - L| \leq \text{ something that only depends on } |x - a|.$$

This kind of inequality is called an ***estimate of*** $|g(x) - L|$ ***in terms of*** $|x - a|$. Next, assume that $|x - a| < \delta$, and turn the estimate you have into one of the form

$$|g(x) - L| \leq \text{ something that only depends on } \delta.$$

Finally, figure out how small δ has to be for the Right Hand Side of this last inequality to be less than ε.

If you can find such a $\delta > 0$ no matter which $\varepsilon > 0$ you were given then you have proved that $\lim_{x\to a} g(x) = L$. If you cannot find such a δ then one of the following has occurred:

(1) there might be a better estimate for $|g(x) - L|$
(2) the limit $\lim_{x\to a} g(x)$ exists, but it isn't L,
(3) the limit $\lim_{x\to a} g(x)$ does not exist.

A common mistake: after using a lot of algebra to estimate $g(x) - L$ you get confused and in the end a find a δ that not only depends on ε, but also on x. The dependence on ε is OK (unavoidable), but ***the dependence on x is not allowed.***

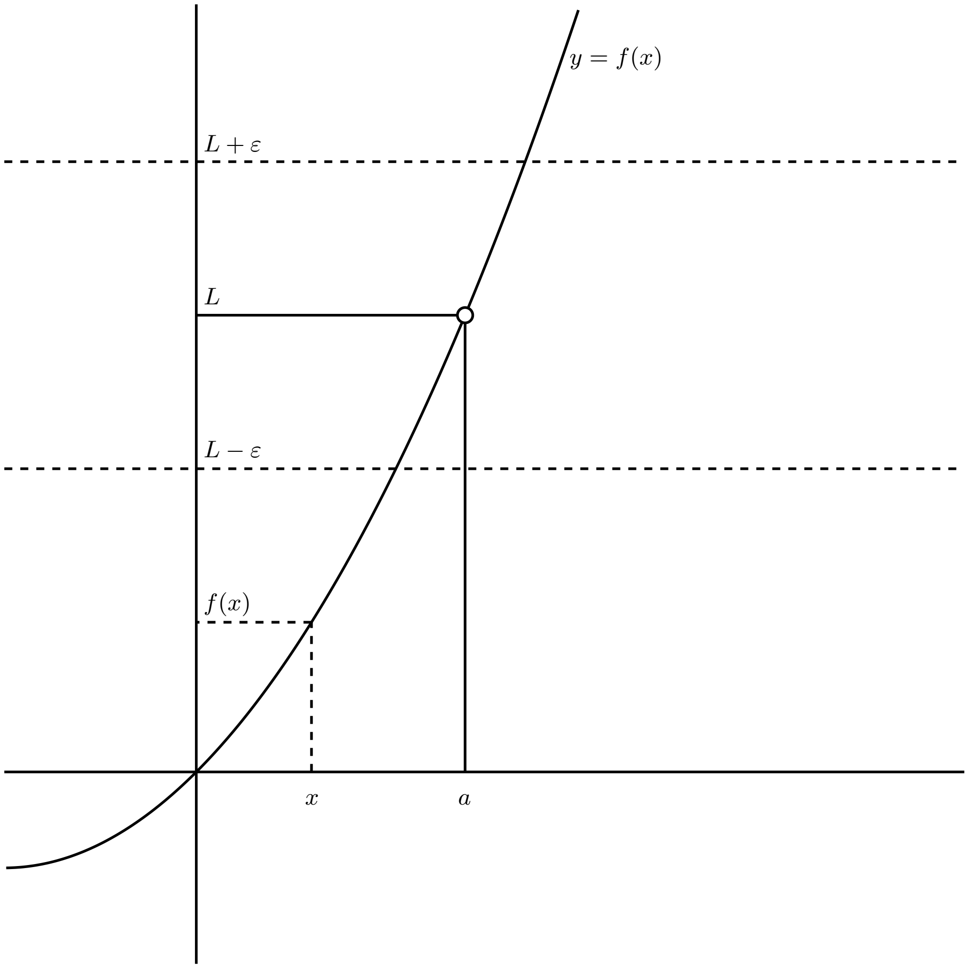


FIGURE 5.
It looks like $\lim_{x\to a} f(x) = L$. How close do you have to choose x to guarantee that $f(x)$ differs less than ε from the apparent limit L?
See the next figure for an answer.

12.1. *Show that* $\lim_{x\to 3} 3x+2 = 11$

We have $g(x) = 3x = 2$, $a = 3$ and $L = 11$. We want to show that $|g(x) - L| < \varepsilon$ follows from $|x - a| < \delta$ provided we choose $\delta > 0$ small enough (depending on ε). To see how small we have to choose δ we look at $|g(x) - L|$

$$|g(x) - L| = \big|(3x+2) - 11\big| = |3x - 9| = 3 \cdot |x - 3| = 3 \cdot |x - a|.$$

So $|x - a| < \delta$ implies $|g(x) - L| < 3\delta$. To guarantee $|g(x) - L| < \varepsilon$ we must therefore choose $3\delta \le \varepsilon$. If we choose $\delta = \frac{1}{3}\varepsilon$, then the conclusion is that for all $x \neq 3$ it follows from $|x - 3| < \delta$ that $|g(x) - L| < \varepsilon$. Therefore $\lim_{x\to 3} g(x) = L$.

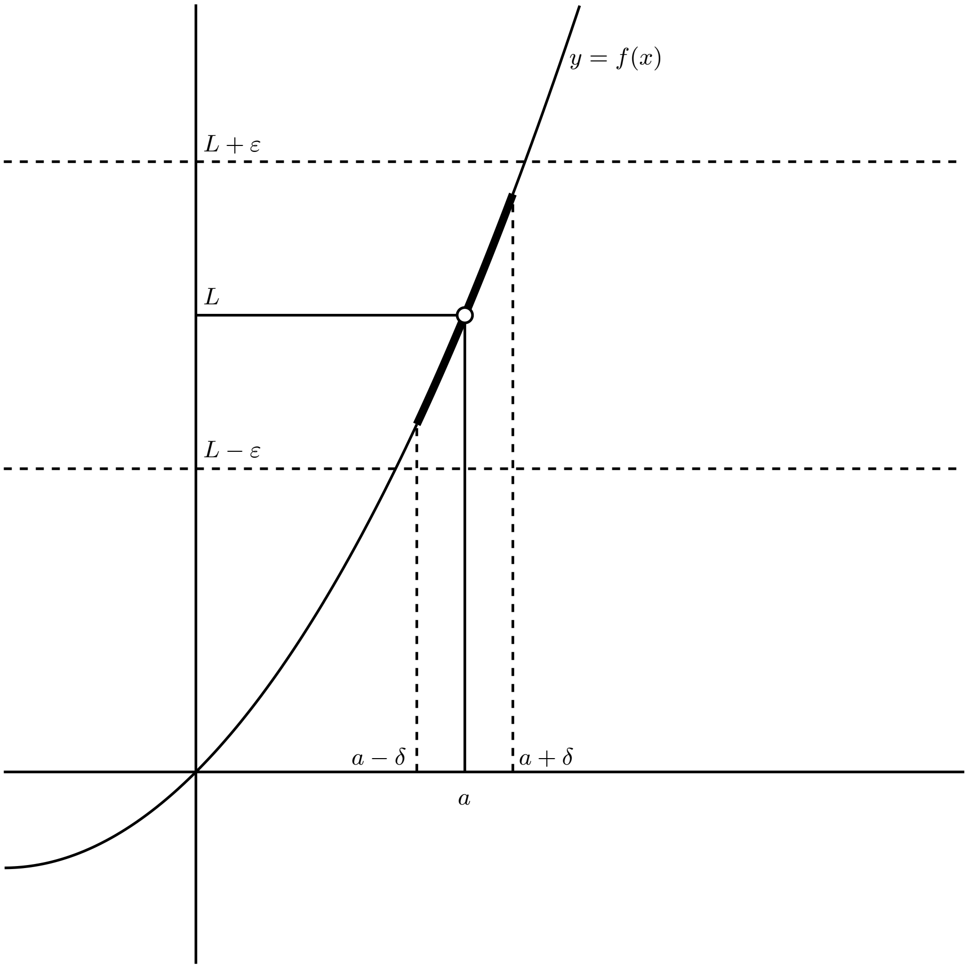


Figure 6.
If you choose δ as in this figure, then $|x - a| < \delta$ implies $|f(x) - L| < \varepsilon$. The same would have still been true if we had chosen a smaller δ, and a slightly larger δ would also work. The important thing is that one can find such a δ.

12.2. *Show that* $\lim_{x\to 1} x^2 = 1$

Here $g(x) = x^2$, $a = 1$ and $L = 1$. We want to show that choosing $|x-1| < \delta$ implies $|x^2 - 1| < \varepsilon$ provided we choose the right δ (which is allowed to depend on ε.)

We begin by estimating the difference $|x^2 - 1|$

$$|x^2 - 1| = |(x-1)(x+1)| = |x-1| \cdot |x+1|.$$

If $|x-1| < \delta$ then $1 - \delta < x < 1 + \delta$ and thus

$$|x+1| < 2 + \delta$$

so that $|x-1| < \delta$ implies

$$|x^2 - 1| < \delta(2 + \delta).$$

We want to make this less than ε by choosing δ small enough. One approach would be to solve the (quadratic) inequality

$$\delta(2 + \delta) < \varepsilon$$

for δ but there is a trick that steers us away from quadratic equations. Namely, if we are going to choose δ small anyway, why not agree to choose $\delta \leq 1$? If we promise never to choose $\delta > 1$, then $|x-1| < \delta$ implies

$$|x^2 - 1| < \delta(2 + \delta) \leq 3\delta.$$

To guarantee that this does not exceed ε we choose $3\delta \leq \varepsilon$, i.e. $\delta = \frac{1}{3}\varepsilon$. This requirement together with our earlier promise to choose $\delta \leq 1$ leads us to this choice of δ

$$\delta = \text{the smaller of } 1 \text{ and } \frac{\varepsilon}{3}.$$

We have shown that if you choose δ this way, then $|x-1| < \delta$ implies $|x^2 - 1| < \varepsilon$, no matter what $\varepsilon > 0$ is.

12.3. *Show that* $\lim_{x\to 4} 1/x = 1/4$

Solution: We apply the definition with $a = 4$, $L = 1/4$ and $g(x) = 1/x$. Thus, for given $\varepsilon > 0$ we try to show that if $|x-4|$ is small enough then one has $|g(x) - 1/4| < \varepsilon$.

We begin by estimating $|g(x) - \frac{1}{4}|$ in terms of $|x-4|$:

$$|g(x) - 1/4| = \left|\frac{1}{x} - \frac{1}{4}\right| = \frac{|x-4|}{|4x|}.$$

This quantity will be small if $|x-4|$ is small, except if division by $4x$ makes the quotient larger. For that to happen x would have to be close to zero (division by a small number gives a large number). By requiring x to be close to 4 we can keep x away from zero. The way to do this is to agree now that we will always take $\delta \leq 3$, no matter what ε is. Then $|x-4| < \delta$ implies

$$|x-4| < \delta \leq 3 \implies 1 < x < 7 \implies |4x| > 4 \implies \frac{1}{|4x|} < \frac{1}{4}$$

and hence,

$$|g(x) - 1/4| = \frac{|x-4|}{|4x|} < \frac{\delta}{4}.$$

Hence if we choose $\delta = 4\varepsilon$ or any smaller number, then $|x-4| < \delta$ implies $|g(x) - 4| < \varepsilon$. Of course we have to honor our agreement never to choose $\delta > 3$, so our choice of δ is

$$\delta = \text{the smaller of } 3 \text{ and } 4\varepsilon.$$

Exercises

12.1 – Use the ε–δ definition to prove that

(i) $\lim_{x\to 1} 2x-4=6$ **(ii)** $\lim_{x\to 2} x^2-7x+3=-7$

(iii) $\lim_{x\to 3} x^3=27$ **(iv)** $\lim_{x\to 0} \sqrt{|x|}=0$

13. Variations on the limit theme

Not all limits are "for $x \to a$." here we describe some possible variations on the concept of limit.

13.1. *Left and right limits*

When we let "x approach a" we allow x to be both larger or smaller than a, as long as x gets close to a. If we explicitly want to study the behaviour of $g(x)$ as x approaches a through values larger than a, then we write

$$\lim_{x\searrow a} g(x) \text{ or } \lim_{x\to a+} g(x) \text{ or } \lim_{x\to a+0} g(x) \text{ or } \lim_{x\to a, x>a} g(x).$$

All four notations are in use. Similarly, to designate the value which $g(x)$ approaches as x approaches a through values below a one writes

$$\lim_{x\nearrow a} g(x) \text{ or } \lim_{x\to a-} g(x) \text{ or } \lim_{x\to a-0} g(x) \text{ or } \lim_{x\to a, x<a} g(x).$$

The precise definition of right limits goes like this:

Definition. 13.1 (Definition of right-limits). *Let g be a function. Then*

$$\lim_{x\searrow a} g(x) = L. \tag{10}$$

means that for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$a < x < a+\delta \implies |g(x)-L| < \varepsilon$$

holds for all x in the domain of g.

The left-limit, i.e. the one-sided limit in which x approaches a through values less than a is defined in a similar way. The following theorem tells you how to use one-sided limits to decide if a function $g(x)$ has a limit at $x = a$.

Theorem 13.2. *If both one-sided limits*

$$\lim_{x\searrow a} g(x) = L_+, \text{ and } \lim_{x\nearrow a} g(x) = L_-$$

exist, then

$$\lim_{x\to a} g(x) \text{ exists} \iff L_+ = L_-.$$

In other words, if a function has both left- and right-limits at some $x = a$, then that function has a limit at $x = a$ if the left- and right-limits are equal.

13.2. *Limits at infinity.*

Instead of letting x approach some finite number, one can let x become "larger and larger" and ask what happens to $g(x)$. If there is a number L such that $g(x)$ gets arbitrarily close to L if one chooses x sufficiently large, then we write

$$\lim_{x\to\infty} g(x) = L, \quad \text{or} \quad \lim_{x\uparrow\infty} g(x) = L, \quad \text{or} \quad \lim_{x\nearrow\infty} g(x) = L.$$

("The limit for x going to infinity is L.")

13.3. *Example – Limit of* $1/x$

The larger you choose x, the smaller its reciprocal $1/x$ becomes. Therefore, it seems reasonable to say

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Here is the precise definition:

Definition. 13.3 (Definition of limit at ∞). *Let g be some function which is defined on some interval $x_0 < x < \infty$. If there is a number L such that for every $\varepsilon > 0$ one can find an A such that*

$$x > A \implies |g(x) - L| < \varepsilon$$

for all x, then we say that the limit of $g(x)$ for $x \to \infty$ is L.

The definition is very similar to the original definition of the limit. Instead of δ which specifies how close x should be to a, we now have a number A which says how large x should be, which is a way of saying "how close x should be to infinity."

13.4. *Example – Limit of* $1/x$ *(again)*

To ***prove*** that $\lim_{x\to\infty} 1/x = 0$ we apply the definition to $g(x) = 1/x$, $L = 0$.

For given $\varepsilon > 0$ we need to show that

$$\left|\frac{1}{x} - L\right| < \varepsilon \text{ for all } x > A \tag{11}$$

provided we choose the right A.

How do we choose A? A is not allowed to depend on x, but it may depend on ε.

If we assume for now that we will only consider positive values of x, then (11) simplifies to

$$\frac{1}{x} < \varepsilon$$

which is equivalent to

$$x > \frac{1}{\varepsilon}.$$

This tells us how to choose A. Given any positive ε, we will simply choose

$$A = \frac{1}{\varepsilon}$$

Then one has $|\frac{1}{x} - 0| = \frac{1}{x} < \varepsilon$ for all $x > A$. Hence we have proved that $\lim_{x\to\infty} 1/x = 0$.

14. Properties of the Limit

The precise definition of the limit is not easy to use, and fortunately we won't use it very often in this class. Instead, there are a number of properties that limits have which allow you to compute them without having to resort to "epsiloncy."

The following properties also apply to the variations on the limit from 13. I.e. the following statements remain true if one replaces each limit by a one-sided limit, or a limit for $x \to \infty$.

Limits of constants and of x. If a and c are constants, then

$$\lim_{x\to a} c = c \tag{P_1}$$

and

$$\lim_{x\to a} x = a. \tag{P_2}$$

Limits of sums, products and quotients. Let F_1 and F_2 be two given functions whose limits for $x \to a$ we know,

$$\lim_{x\to a} F_1(x) = L_1, \qquad \lim_{x\to a} F_2(x) = L_2.$$

Then

$$(P_3) \qquad \lim_{x\to a}\bigl(F_1(x) + F_2(x)\bigr) = L_1 + L_2,$$

$$(P_4) \qquad \lim_{x\to a}\bigl(F_1(x) - F_2(x)\bigr) = L_1 - L_2,$$

$$(P_5) \qquad \lim_{x\to a}\bigl(F_1(x) \cdot F_2(x)\bigr) = L_1 \cdot L_2$$

Finally, if $\lim_{x\to a} F_2(x) \neq 0$,

$$(P_6) \qquad \lim_{x\to a} \frac{F_1(x)}{F_2(x)} = \frac{L_1}{L_2}.$$

In other words the limit of the sum is the sum of the limits, etc. One can prove these laws using the definition of limit in §12 but we will not do this here. However, I hope these laws seem like common sense: if, for x close to a, the quantity $F_1(x)$ is close to L_1 and $F_2(x)$ is close to L_2, then certainly $F_1(x) + F_2(x)$ should be close to $L_1 + L_2$.

There are two more properties of limits which we will add to this list later on. They are the "Sandwich Theorem" (§18) and the substitution theorem (§19).

15. Examples of limit computations

15.1. *Find* $\lim_{x\to 2} x^2$

One has

$$\begin{aligned}\lim_{x\to 2} x^2 &= \lim_{x\to 2} x \cdot x \\ &= \bigl(\lim_{x\to 2} x\bigr) \cdot \bigl(\lim_{x\to 2} x\bigr) && \text{by } (P_5) \\ &= 2 \cdot 2 = 4.\end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{x\to 2} x^3 &= \lim_{x\to 2} x \cdot x^2 \\ &= \bigl(\lim_{x\to 2} x\bigr) \cdot \bigl(\lim_{x\to 2} x^2\bigr) && (P_5) \text{ again} \\ &= 2 \cdot 4 = 8,\end{aligned}$$

and, by (P_4)

$$\lim_{x\to 2} x^2 - 1 = \lim_{x\to 2} x^2 - \lim_{x\to 2} 1 = 4 - 1 = 3,$$

and, by (P_4) again,

$$\lim_{x\to 2} x^3 - 1 = \lim_{x\to 2} x^3 - \lim_{x\to 2} 1 = 8 - 1 = 7,$$

Putting all this together, one gets

$$\lim_{x\to 2} \frac{x^3 - 1}{x^2 - 1} = \frac{2^3 - 1}{2^2 - 1} = \frac{8 - 1}{4 - 1} = \frac{7}{3}$$

because of (P_6). To apply (P_6) we must check that the denominator ("L_2")is not zero. Since the denominator is 3 everything is OK, and we were allowed to use (P_6).

15.2. *Try the examples 11.2 and 11.3 using the limit properties*

To compute $\lim_{x\to 2}(x^2-2x)(x^2-4)$ we first use the limit properties to find

$$\lim_{x\to 2} x^2-2x = 0 \text{ and } \lim_{x\to 2} x^2-4=0.$$

to complete the computation we would like to apply the last property (P_6) about quotients, but this would give us

$$\lim_{x\to 2} g(x) = \frac{0}{0}.$$

The denominator is zero, so we were not allowed to use (P_6) (and the result doesn't mean anything anyway). We have to do something else.

The function we are dealing with is a ***rational function***, which means that its the quotient of two polynomials. For such functions there is an algebra trick which always allows you to compute the limit even if you first get $\frac{0}{0}$. The thing to do is to divide numerator and denominator by $x-2$. In our case we have

$$x^2-2x=(x-2)\cdot x, \qquad x^2-4=(x-2)\cdot(x+2)$$

so that

$$\lim_{x\to 2} g(x) = \lim_{x\to 2}\frac{(x-2)\cdot x}{(x-2)\cdot(x+2)} = \lim_{x\to 2}\frac{x}{x+2}.$$

After this simplification we ***can*** use the properties ($P_{\dots}$) to compute

$$\lim_{x\to 2} g(x) = \frac{2}{2+2} = \frac{1}{2}.$$

The point in the above example is that

$$\frac{x^2-2x}{x^2-4} = \frac{x}{x+2}$$

for $x \neq 2$ and the right hand side (but not the left) is meaningful even when $x=2$.

15.3. *Example – Find* $\lim_{x\to 2}\sqrt{x}$

There is nothing in the limit properties which tells us how to deal with a square root, and using them we can't prove that there is a limit. However, if you assume that the limit exists, i.e. that there is a number L for which $\sqrt{x}$ gets "closer and closer" to L as x "approaches 2," then the limit properties allow us to find this the limit L.

So, suppose that there is a number L with

$$\lim_{x\to 2}\sqrt{x} = L.$$

Then property (P_5) implies that

$$L^2 = \big(\lim_{x\to 2}\sqrt{x}\big)\cdot\big(\lim_{x\to 2}\sqrt{x}\big) = \lim_{x\to 2}\sqrt{x}\cdot\sqrt{x} = \lim_{x\to 2} x = 2.$$

In other words, $L^2=2$, and hence L must be either $\sqrt{2}$ or $-\sqrt{2}$. We can reject the latter because whatever x does, its squareroot is always a positive number, and hence it can never "get close to" a negative number like $-\sqrt{2}$.

Our conclusion: if the limit exists, then

$$\lim_{x\to 2}\sqrt{x} = \sqrt{2}.$$

The result is not surprising: if x gets close to 2 then $\sqrt{x}$ gets close to $\sqrt{2}$.

15.4. *Example – Find* $\lim_{x\to 2}\sqrt{x}$

This is the same question as in the previous example. There we showed how the limit properties imply that the only possible value of the limit is $\sqrt{2}$, but we didn't show that $\sqrt{2}$ actually is the limit. For that one has to wield εs and δs. Here's how:

Assuming $|x-2|<\delta$ we estimate the difference $|\sqrt{x}-\sqrt{2}|$ as follows

$$|\sqrt{x}-\sqrt{2}| = \left|\frac{x-2}{\sqrt{x}+\sqrt{2}}\right| = \frac{|x-2|}{\sqrt{x}+\sqrt{2}}.$$

Since $\sqrt{x}\geq 0$ for all x we have $\sqrt{x}+\sqrt{2}\geq\sqrt{2}$, and thus if $|x-2|\leq\delta$

$$|\sqrt{x}-\sqrt{2}| \leq \frac{|x-2|}{\sqrt{2}} < \frac{\delta}{\sqrt{2}}.$$

If we want $|\sqrt{x}-\sqrt{2}|<\varepsilon$, then we should choose $\frac{\delta}{\sqrt{2}}\leq\varepsilon$. Thus with

$$\delta = \varepsilon\sqrt{2}$$

we find that $|x-2|<\delta$ always implies $|\sqrt{x}-\sqrt{2}|<\varepsilon$, no matter which $\varepsilon>0$ is given.

15.5. *Example – The derivative of $\sqrt{x}$ at $x=2$.*

Find

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$$

assuming the result from the previous example.

Solution: The function is a rational function whose numerator and denominator vanish when $x=2$, i.e. the limit is of the form $\frac{0}{0}$. We use the same algebra trick as before, namely we factor numerator and denominator:

$$\frac{\sqrt{x}-\sqrt{2}}{x-2} = \frac{\sqrt{x}-\sqrt{2}}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})} = \frac{1}{\sqrt{x}+\sqrt{2}}.$$

Now one can use the limit properties to compute

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

15.6. *Limit as $x\to\infty$ of rational functions*

A rational function is the quotient of two polynomials, so

$$R(x) = \frac{a_nx^n+\cdots+a_1x+a_0}{b_mx^m+\cdots+b_1x+b_0}. \tag{12}$$

We have seen that

$$\lim_{x\to\infty}\frac{1}{x}=0$$

We even proved this in example 13.4. Using this you can find the limit at ∞ for any rational function $R(x)$ as in (12). One could turn the outcome of the calculation of $\lim_{x\to\infty}R(x)$ into a recipe/formula involving the degrees n and m of the numerator and denominator, and also their coefficients a_i, b_j, which students would then memorize, but it is better to remember "the trick."

To find $\lim_{x\to\infty}R(x)$ divide numerator and denominator by x^m (the highest power of x occurring in the denominator).

For example, let's compute

$$\lim_{x\to\infty}\frac{3x^2+3}{5x^2+7x-39}.$$

Remember the trick and divide top and bottom by x^2, and you get

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^2+3}{5x^2+7x-39} &= \lim_{x\to\infty}\frac{3+3/x^2}{5+7/x-39/x^2}\\ &= \frac{\lim_{x\to\infty}3+3/x^2}{\lim_{x\to\infty}5+7/x-39/x^2}\\ &= \frac{3}{5}\end{aligned}$$

Here we have used the limit properties (P_*) to break the limit down into little pieces like $\lim_{x\to\infty}39/x^2$ which we can compute as follows

$$\lim_{x\to\infty}39/x^2 = \lim_{x\to\infty}39\cdot\left(\frac{1}{x}\right)^2 = \Big(\lim_{x\to\infty}39\Big)\cdot\left(\lim_{x\to\infty}\frac{1}{x}\right)^2 = 39\cdot 0^2 = 0.$$

15.7. *Another example with a rational function*

Compute

$$\lim_{x\to\infty}\frac{x}{x^3+5}.$$

We apply "the trick" again and divide numerator and denominator by x^3. This leads to

$$\lim_{x\to\infty}\frac{x}{x^3+5} = \lim_{x\to\infty}\frac{1/x^2}{1+5/x^3} = \frac{\lim_{x\to\infty}1/x^2}{\lim_{x\to\infty}1+5/x^3} = \frac{0}{1} = 0.$$

To show all possible ways a limit of a rational function can turn out we should do yet another example, but that one belongs in the next section (see example 16.5.)

16. **When limits fail to exist**

In the last couple of examples we worried about the possibility that a limit $\lim_{x\to a}g(x)$ actually might not exist. This can actually happen, and in this section we'll see a few examples of what failed limits look like. First let's agree on what we will call a "failed limit."

Definition. 16.1. *If there is no number L such that $\lim_{x\to a}g(x) = L$, then we say that the limit $\lim_{x\to a}g(x)$ does not exist.*

16.1. *The sign function near $x = 0$*

The "sign function" is defined[2]

$$\operatorname{sign}(x) = \begin{cases}-1 & \text{for } x<0\\ 0 & \text{for } x=0\\ 1 & \text{for } x>0\end{cases}$$

Note that the sign of zero is zero. But does the sign function has a limit at $x = 0$, i.e. does

$$\lim_{x\to 0}\operatorname{sign}(x)$$

exist? And is it also zero? The answer is ***no*** and ***no***, and here is why: suppose that for some number L one had

$$\lim_{x\to 0}\operatorname{sign}(x) = L,$$

---

[2] Some people don't like the notation $\operatorname{sign}(x)$, and prefer to write

$$g(x) = \frac{x}{|x|}$$

instead of $g(x) = \operatorname{sign}(x)$. If you think about this formula for a moment you'll see that $\operatorname{sign}(x) = x/|x|$ for all $x \neq 0$. When $x = 0$ the quotient $x/|x|$ is of course not defined.

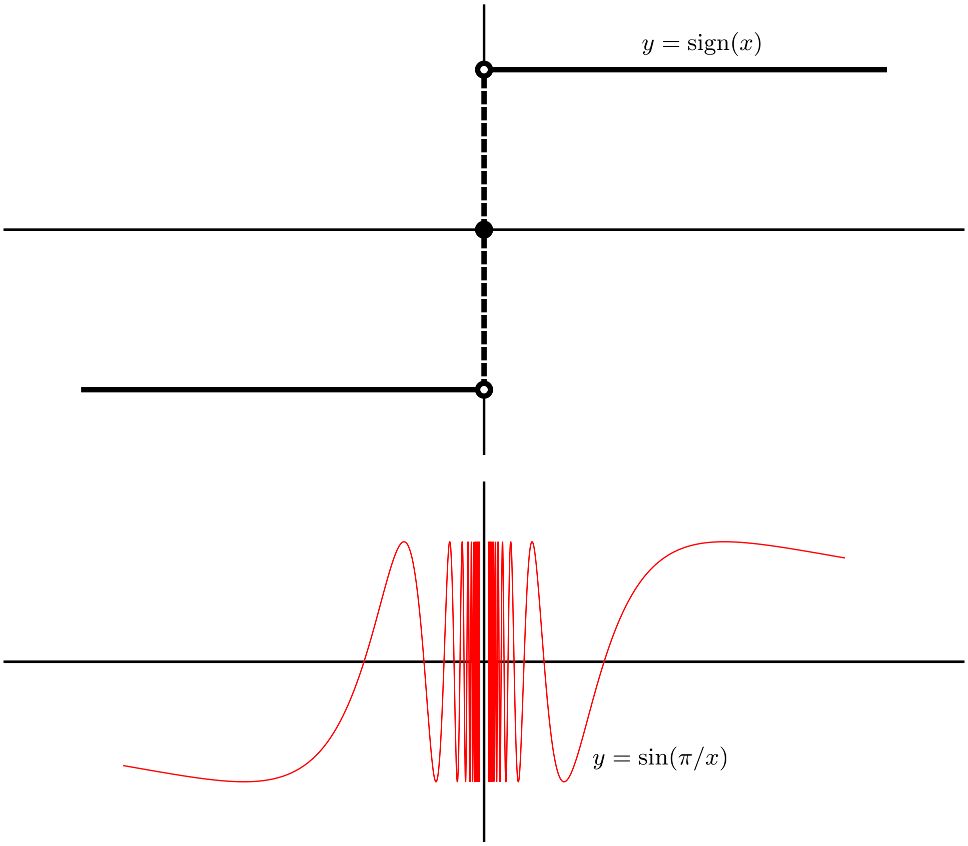
then since for arbitrary small positive values of x one has $\operatorname{sign}(x) = +1$ one would think that $L = +1$. But for arbitrarily small negative values of x one has $\operatorname{sign}(x) = -1$, so one would conclude that $L = -1$. But one number L can't be both $+1$ and -1 at the same time, so there is no such L, i.e. there is no limit.

$$\lim_{x\to 0} \operatorname{sign}(x) \text{ does not exist.}$$

In this example the one-sided limits do exist, namely,

$$\lim_{x\searrow 0} \operatorname{sign}(x) = 1 \text{ and } \lim_{x\nearrow 0} \operatorname{sign}(x) = -1.$$

All this says is that when x approaches 0 through positive values, its sign approaches $+1$, while if x goes to 0 through negative values, then its sign approaches -1.



Graphs of $g(x) = \operatorname{sign}(x)$ and $g(x) = \sin(1/x)$
for $-3 < x < 3, \ \ x \neq 0$.

16.2. *The example of the backward sine*

Contemplate the limit as $x \to 0$ of the "backward sine," i.e.

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right).$$

When $x = 0$ the function $g(x) = \sin(1/x)$ is not defined, because its definition involves division by x. What happens to $g(x)$ as $x \to 0$? First, $1/x$ becomes larger and larger ("goes to infinity") as $x \to 0$. Then, taking the sine, we see that $\sin(1/x)$ oscillates between +1 and −1 infinitely often as $x \to 0$. This means that $g(x)$ gets close to any number between −1 and +1 as $x \to 0$, but that the function $g(x)$ ***never stays close*** to any particular value because it keeps oscillating up and down.

Here again, the limit $\lim_{x\to 0} g(x)$ does not exist. We have arrived at this conclusion by only considering what $g(x)$ does for small positive values of x. So the limit fails to exist in a stronger way than in the example of the sign-function. There, even though the limit didn't exist, the one-sided limits existed. In the present example we see that even the one-sided limit

$$\lim_{x\searrow 0} \sin\frac{1}{x}$$

does not exist.

16.3. *Trying to divide by zero using a limit*

The expression 1/0 is not defined, but what about

$$\lim_{x\to 0}\frac{1}{x}?$$

This limit also does not exist. Here are two reasons:

It is common wisdom that if you divide by a small number you get a large number, so if you divide 1 by "smaller and smaller" numbers, the result will get "larger and larger." In particular, it will not be able to stay close to any particular finite number. So the limit can't exist.

"Common wisdom" is not always a reliable tool in mathematical proofs, so here is a better argument. The limit can't exist, because that would contradict the limit properties $(P_1)\cdots(P_6)$. Namely, suppose that there were an number L such that

$$\lim_{x\to 0}\frac{1}{x} = L.$$

Then the limit property (P_5) would imply that

$$\lim_{x\to 0}\big(\frac{1}{x}\cdot x\big) = \big(\lim_{x\to 0}\frac{1}{x}\big)\cdot\big(\lim_{x\to 0} x\big) = L\cdot 0 = 0.$$

On the other hand $\frac{1}{x}\cdot x = 1$ so the above limit should be 1! A number can't be both 0 and 1 at the same time, so we have a contradiction. The assumption that $\lim_{x\to 0} 1/x$ exists is to blame, so it must go.

16.4. *Using limit properties to show a limit does **not** exist*

The limit properties tell us how to prove that certain limits exist (and how to compute them). Although it is perhaps not so obvious at first sight, they also allow you to prove that certain limits do not exist. The previous example shows one instance of such use. Here is another.

Property (P_3) says that if both $\lim_{x\to a} g(x)$ and $\lim_{x\to a} h(x)$ exist then $\lim_{x\to a} g(x) + h(x)$ also must exist. You can turn this around and say that if $\lim_{x\to a} g(x)+h(x)$ does not exist then either $\lim_{x\to a} g(x)$ or $\lim_{x\to a} h(x)$ does not exist (or both limits fail to exist).

For instance, the limit

$$\lim_{x\to 0}\frac{1}{x} - x$$

can't exist, for if it did, then the limit

$$\lim_{x\to 0}\frac{1}{x} = \lim_{x\to 0}\big(\frac{1}{x} - x + x\big) = \lim_{x\to 0} bigl(\frac{1}{x} - x\big) + \lim_{x\to 0} x$$

would also have to exist, and we know $\lim_{x\to 0}\frac{1}{x}$ doesn't exist.

16.5. *Limits at ∞ which don't exist*

If you let x go to ∞, then x will not get "closer and closer" to any particular number L, so it seems reasonable to guess that

$$\lim_{x\to\infty} x \text{ does not exist.}$$

One can prove this from the limit definition (and see exercise 2).

Let's consider

$$L = \lim_{x\to\infty} \frac{x^2+2x-1}{x+2}.$$

Once again we divide numerator and denominator by the highest power in the denominator (i.e. x)

$$L = \lim_{x\to\infty} \frac{x+2-\frac{1}{x}}{1+2/x}$$

Here the denominator has a limit ('tis 1), but the numerator does not, for if $\lim_{x\to\infty} x + 2 - \frac{1}{x}$ existed then, since $\lim_{x\to\infty}(2-1/x) = 2$ exists,

$$\lim_{x\to\infty} x = \lim_{x\to\infty} \left[\bigl(x+2-\frac{1}{x}\bigr) - \bigl(2-\frac{1}{x}\bigr)\right]$$

would also have to exist, and $\lim_{x\to\infty} x$ doesn't exist.

So we see that L is the limit of a fraction in which the denominator has a limit, but the numerator does not. In this situation the limit L itself can never exist. If it did, then

$$\lim_{x\to\infty} \bigl(x+2-\frac{1}{x}\bigr) = \lim_{x\to\infty} \frac{x+2-\frac{1}{x}}{1+2/x}\cdot(1+2/x)$$

would also have to have a limit.

17. **What's in a name?**

There is a big difference between the variables x and a in the formula

$$\lim_{x\to a} 2x+1,$$

namely a is a ***free variable***, while x is a ***dummy variable*** (or "placeholder" or a "bound variable.")

The difference between these two kinds of variables is this:

- if you replace a dummy variable in some formula consistently by some other variable then the value of the formula does not change. On the other hand, it never makes sense to substitute a number for a dummy variable.
- the value of the formula may depend on the value of the free variable.

To understand what this means consider the example $\lim_{x\to a} 2x+1$ again. The limit is easy to compute:

$$\lim_{x\to a} 2x+1 = 2a+1.$$

If we replace x by, say u (systematically) then we get

$$\lim_{u\to a} 2u+1$$

which is again equal to $2a+1$. This computation says that *if some number gets close to a then two times that number plus one gets close to* $2a+1$. This is a very wordy way of expressing the formula, and you can shorten things by giving a name (like x or u) to the number which approaches a. But the result of our computation shouldn't depend on the name we choose, i.e. it doesn't matter if we call it x or u.

Since the name of the variable x doesn't matter it is called a dummy variable. Some prefer to call x a bound variable, meaning that in

$$\lim_{x\to a} 2x+1$$

the x in the expression $2x+1$ is bound to the x written underneath the limit – you can't change one without changing the other.

Substituting a number for a dummy variable usually leads to complete nonsense. For instance, let's try setting $x = 3$ in our limit, i.e. what is

$$\lim_{3\to a} 2\cdot 3+1\,?$$

Of course $2\cdot 3+1 = 7$, but what does 7 do when 3 gets closer and closer to the number a? That's a silly question, because 3 is a constant and it doesn't "get closer" to some other number like a! If you ever see 3 get closer to another number then it's time to take a vacation.

On the other hand the variable a is free: you can assign it particular values, and its value will affect the value of the limit. For instance, if we set $a = 3$ (but leave x alone) then we get

$$\lim_{x\to 3} 2x+1$$

and there's nothing strange about that (the limit is $2\cdot 3+1 = 7$, no problem.) You could substitute other values of a and you would get a different answer. In general you get $2a+1$.

18. Limits and Inequalities

This section has two theorems which let you compare limits of different functions. The properties in these theorems are not formulas that allow you to compute limits like the properties $(P_1)\ldots(P_6)$ from §14. Instead, they allow you to ***reason*** about limits, i.e. they let you say that this or that limit is positive, or that it must be the same as some other limit which you find easier to think about.

The first theorem should not surprise you – all it says is that bigger functions have bigger limits.

Theorem 18.1. *Let f and g be functions whose limits for $x\to a$ exist, and assume that $f(x)\le g(x)$ holds for all x. Then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

A useful special case arises when you set $f(x) = 0$. The theorem then says that if a function g never has negative values, then its limit will also never be negative.

The statement may seem obvious, but it still needs a proof, starting from the ε-δ definition of limit. This will be done in lecture.

Here is the second theorem about limits and inequalities.

Theorem 18.2 (The Sandwich Theorem). *Suppose that*

$$f(x)\le g(x)\le h(x)$$

(for all x) and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x).$$

Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x).$$

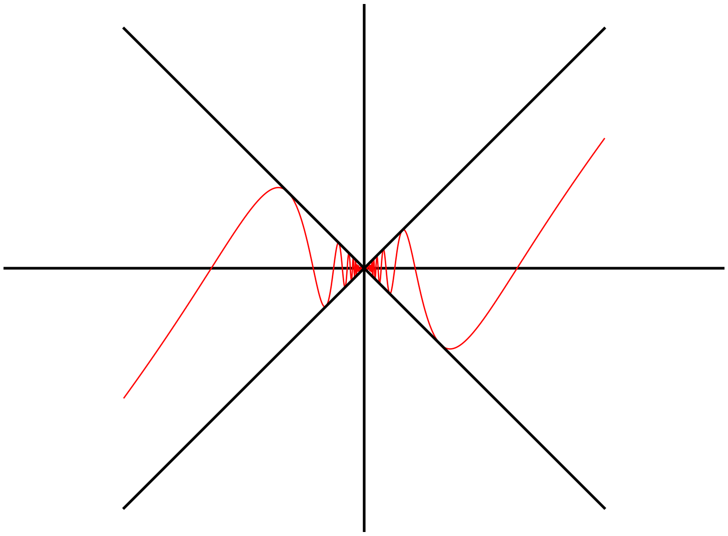


Figure 7. Graphs of $|x|$, $-|x|$ and $x\cos(1/x)$ for $-1.2 < x < 1.2$

The theorem is useful when you want to know the limit of g, and when you can ***sandwich*** it between two functions f and h whose limits are easier to compute. The Sandwich Theorem looks like the first theorem of this section, but there is an important difference: in the Sandwich Theorem you don't have to assume that the limit of g exists. The inequalities $f \le g \le h$ combined with the circumstance that f and h have the same limit are enough to guarantee that the limit of g exists.

18.1. *A backward cosine sandwich*

The Sandwich Theorem says that if the function $g(x)$ is sandwiched between two functions $f(x)$ and $h(x)$ and the limits of the outside functions f and h exist and are equal, then the limit of the inside function g exists and equals this common value. For example

$$-|x| \le x\cos\frac{1}{x} \le |x|$$

since the cosine is always between -1 and 1. Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$$

the sandwich theorem tells us that

$$\lim_{x\to 0} x\cos\frac{1}{x} = 0.$$

Note that the limit $\lim_{x\to 0}\cos(1/x)$ does ***not*** exist, for the same reason that the "backward sine" did not have a limit for $x \to 0$ (see example 16.2). Multiplying with x changed that.

19. **Continuity**

Definition. 19.1. *A function g is **continuous** at a if*

$$\lim_{x\to a} g(x) = g(a) \tag{13}$$

A function is continuous if it is continuous at every a in its domain.

Note that when we say that a function is continuous on some interval it is understood that the domain of the function includes that interval. For example, the function $f(x) = 1/x^2$ is continuous on the interval $1 < x < 5$ but is ***not*** continuous on the interval $-1 < x < 1$.

19.1. *Polynomials are continuous*

For instance, let us show that $P(x) = x^2 + 3x$ is continuous at $x = 2$. To show that you have to prove that

$$\lim_{x\to 2} P(x) = P(2),$$

i.e.

$$\lim_{x\to 2} x^2 + 3x = 2^2 + 3 \cdot 2.$$

You can do this two ways: using the definition with ε and δ (i.e. the hard way), or using the limit properties $(P_1)\dots(P_6)$ from §14 (just as good, and easier, even though it still takes a few lines to write it out – do both!)

19.2. *Rational functions are continuous*

Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, and let a be any number in the domain of R, i.e. any number for which $Q(a) \neq 0$. Then one has

$$\begin{aligned}
\lim_{x\to a} R(x) &= \lim_{x\to a} \frac{P(x)}{Q(x)} \\
&= \frac{\lim_{x\to a} P(x)}{\lim_{x\to a} Q(x)} && \text{property } (P_6) \\
&= \frac{P(a)}{Q(a)} && P \text{ and } Q \text{ are continuous} \\
&= R(a).
\end{aligned}$$

This shows that R is indeed continuous at a.

19.3. *Some discontinuous functions*

If $\lim_{x\to a} g(x)$ does not exist, then it certainly cannot be equal to $g(a)$, and therefore any failed limit provides an example of a discontinuous function.

For instance, the sign function $g(x) = \operatorname{sign}(x)$ from example **??** is not continuous at $x = 0$.

Is the backward sine function $g(x) = \sin(1/x)$ from example 16.2 also discontinuous at $x = 0$? No, it is not, for two reasons: first, the limit $\lim_{x\to 0} \sin(1/x)$ does not exist, and second, we haven't even defined the function $g(x)$ at $x = 0$, so even if the limit existed, we would have no value $g(0)$ to compare it with.

19.4. *How to make functions discontinuous*

Here is a discontinuous function:

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 3, \\ 47 & \text{if } x = 3. \end{cases}$$

In other words, we take a continuous function like $g(x) = x^2$, and change its value somewhere, e.g. at $x = 3$. Then

$$\lim_{x\to 3} f(x) = 9 \neq 47 = f(3).$$

The reason that the limit is 9 is that our new function $f(x)$ coincides with our old continuous function $g(x)$ for all x except $x = 3$. Therefore the limit of $f(x)$ as $x \to 3$ is the same as the limit of $g(x)$ as $x \to 3$, and since g is continuous this is $g(3) = 9$.

19.5. *Sandwich in a bow tie*

We return to the function from example 18.1. Consider

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

Then f is continuous at $x = 0$ by the Sandwich Theorem (see Example 18.1).

If we change the definition of f by picking a different value at $x = 0$ the new function will not be continuous, since changing f at $x = 0$ does not change the limit $\lim_{x\to 0} f(x)$. Since this limit is zero, $f(0) = 0$ is the only possible choice of $f(0)$ which makes f continuous at $x = 0$.

20. **Substitution in Limits**

Given two functions f and g one can consider their composition $h(x) = f(g(x))$. To compute the limit

$$\lim_{x\to a} f\big(g(x)\big)$$

we write $u = g(x)$, so that we want to know

$$\lim_{x\to a} f(u) \text{ where } u = g(x).$$

Suppose that you can find the limits

$$L = \lim_{x\to a} g(x) \text{ and } \lim_{u\to L} f(u) = M.$$

Then it seems reasonable that as x approaches a, $u = g(x)$ will approach L, and $f(g(x))$ approaches M.

This is in fact a theorem:

Theorem 20.1. *If* $\lim_{x\to a} g(x) = a$, *and if the function* f *is continuous at* $u = L$, *then*

$$\lim_{x\to a} f\big(g(x)\big) = \lim_{u\to L} f(u) = f(L).$$

Another way to write this is

$$\lim_{x\to a} f\big(g(x)\big) = f\big(\lim_{x\to a} g(x)\big).$$

20.1. *Compute* $\lim_{x\to 3} \sqrt{x^3 - 3x^2 + 2}$

The given function is the composition of two functions, namely

$$\sqrt{x^3 - 3x^2 + 2} = \sqrt{u}, \text{ with } u = x^3 - 3x^2 + 2,$$

or, in function notation, we want to find $\lim_{x\to 3} h(x)$ where

$$h(x) = f(g(x)), \text{ with } g(x) = x^3 - 3x^2 + 2 \text{ and } g(x) = \sqrt{x}.$$

Either way, we have

$$\lim_{x\to 3} x^3 - 3x^2 + 2 = 2 \quad \text{and} \quad \lim_{u\to 2} \sqrt{u} = \sqrt{2}.$$

You get the first limit from the limit properties $(P_1)\dots(P_5)$. The second limit says that taking the square root is a continuous function, which it is. We have not proved that (yet), but this particular limit is the one from example 15.3. Putting these two limits together we conclude that the limit is $\sqrt{2}$.

Normally, you write this whole argument as follows:

$$\lim_{x\to 3}\sqrt{x^3-3x^2+2}=\sqrt{\lim_{x\to 3}x^3-3x^2+2}=\sqrt{2},$$

where you must point out that $f(x)=\sqrt{x}$ is a continuous function to justify the first step.

Another possible way of writing this is

$$\lim_{x\to 3}\sqrt{x^3-3x^2+2}=\lim_{u\to 2}\sqrt{u}=\sqrt{2},$$

where you must say that you have substituted $u=x^3-3x^2+2$.

Exercises

20.1 – Find the following limits.

(i) $\lim_{x\to -7}(2x+5)$ **(ii)** $\lim_{x\to 7-}(2x+5)$ **(iii)** $\lim_{x\to -\infty}(2x+5)$

(iv) $\lim_{x\to -4}(x+3)^{2006}$ **(v)** $\lim_{x\to -4}(x+3)^{2007}$ **(vi)** $\lim_{x\to -\infty}(x+3)^{2007}$

(vii) $\displaystyle\lim_{t\to 1}\frac{t^2+t-2}{t^2-1}$ **(viii)** $\displaystyle\lim_{t\nearrow 1}\frac{t^2+t-2}{t^2-1}$ **(ix)** $\displaystyle\lim_{t\to -1}\frac{t^2+t-2}{t^2-1}$

(x) $\displaystyle\lim_{x\to\infty}\frac{x^2+3}{x^2+4}$ **(xi)** $\displaystyle\lim_{x\to\infty}\frac{x^5+3}{x^2+4}$ **(xii)** $\displaystyle\lim_{x\to\infty}\frac{x^2+1}{x^5+2}$

(xiii) $\displaystyle\lim_{x\to\infty}\frac{(2x+1)^4}{(3x^2+1)^2}$ **(xiv)** $\displaystyle\lim_{u\to\infty}\frac{(2u+1)^4}{(3u^2+1)^2}$ **(xv)** $\displaystyle\lim_{t\to 0}\frac{(2t+1)^4}{(3t^2+1)^2}$

20.2 – In the text we proved that $\lim_{x\to\infty}\frac{1}{x}=0$. Show that this implies that $\lim_{x\to\infty}x$ does not exist. Hint: Suppose $\lim_{x\to\infty}x=L$ for some number L. Apply the limit properties to $\lim_{x\to\infty}x\cdot\dfrac{1}{x}$.

20.3 – Evaluate $\displaystyle\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$. Hint: Multiply top and bottom by $\sqrt{x}+3$.

20.4 – Evaluate $\displaystyle\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$.

20.5 – Evaluate $\displaystyle\lim_{x\to 2}\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{2}}}{x-2}$.

20.6 – A function f is defined by

$$f(x)=\begin{cases}x^3 & \text{for } x<-1\\ ax+b & \text{for } -1\le x<1\\ x^2+2 & \text{for } x\ge 1.\end{cases}$$

where a and b are constants. The function f is continuous. What are a and b?

21. Two Limits in Trigonometry

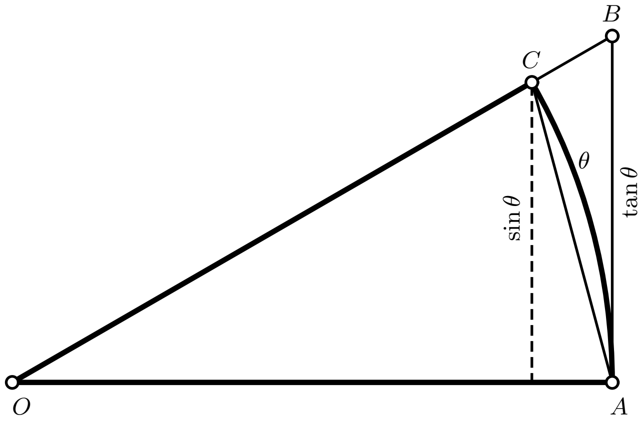
In this section we'll derive a few limits involving the trigonometric functions. You can think of them as saying that for small angles θ one has

$$\sin\theta\approx\theta \quad\text{and}\quad \cos\theta\approx 1-\tfrac{1}{2}\theta^2.$$

We will use these limits when we compute the derivatives of Sine, Cosine and Tangent.

Theorem 21.1.

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$



The circular wedge OAC contains the triangle OAC
and is contained in the right triangle OAB.
The area of triangle OAC is $\frac{1}{2}\sin\theta$. The area of circular wedge OAC is $\frac{1}{2}\theta$.
The area of right triangle OAB is $\frac{1}{2}\tan\theta$.
Hence one has $\sin\theta < \theta < \tan\theta$ for all angles $0 < \theta < \pi/2$.

FIGURE 8. Proof that $\sin\theta \approx \theta$ for small values of θ

Proof. The proof requires a few sandwiches and some geometry.

We begin by only considering positive angles, and in fact we will only consider angles $0 < \theta < \pi/2$. By comparing areas in the drawing we see that for such angles one always has

$$\sin\theta < \theta < \tan\theta. \tag{14}$$

Since $\sin\theta > 0$ for $0 < \theta < \pi/2$ we get

$$0 < \sin\theta < \theta$$

for $0 < \theta < \pi/2$. As $\theta \searrow 0$ both 0 and θ go to zero, so the Sandwich Theorem implies that

$$\lim_{\theta\searrow 0} \sin\theta = 0.$$

Hence

$$\lim_{\theta\searrow 0} \cos\theta = \lim_{\theta\searrow 0} \sqrt{1-\sin^2\theta} = 1.$$

Finally we go back to the first sandwich (14) and divide it by θ

$$\frac{\sin\theta}{\theta} < 1 < \frac{\tan\theta}{\theta} = \frac{1}{\cos\theta}\,\frac{\sin\theta}{\theta}.$$

This implies

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

The Sandwich Theorem can be used once again, and now it gives

$$\lim_{\theta\searrow 0} \frac{\sin\theta}{\theta} = 1.$$

This is a one-sided limit. To get the limit in which $\theta \nearrow 0$, you use that $\sin\theta$ is an odd function. $\square$

Theorem 21.2.

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\frac{1}{2}.$$

Proof. This follows from $\sin^2\theta+\cos^2\theta=1$. Namely,

$$\begin{aligned}\frac{1-\cos\theta}{\theta^2}&=\frac{1}{1+\cos\theta}\,\frac{1-\cos^2\theta}{\theta^2}\\&=\frac{1}{1+\cos\theta}\,\frac{\sin^2\theta}{\theta^2}\\&=\frac{1}{1+\cos\theta}\left\{\frac{\sin\theta}{\theta}\right\}^2.\end{aligned}$$

We have just shown that $\cos\theta\to 1$ and $\frac{\sin\theta}{\theta}\to 1$ as $\theta\to 0$, so the theorem follows. ☐

Exercises

21.1 – Find the limit or show that it does not exist. Distinguish between limits which are infinite and limits which do not exist.

(a) $\lim_{\theta\to 0}\dfrac{\tan\theta}{\theta}$. (b) $\lim_{x\to 0}\dfrac{1-\cos x}{x\sin x}$. (c) $\lim_{x\to\infty}\dfrac{2x^3+3x^2\cos x}{(x+2)^3}$.

(d) $\lim_{x\to 0}\dfrac{\sin(x^2)}{x^2}$. (e) $\lim_{x\to 0}\dfrac{x(1-\cos x)}{\tan^3 x}$. (f) $\lim_{x\to 0}\dfrac{\sin(x^2)}{1-\cos x}$.

(g) $\lim_{x\to 0}\dfrac{\cos x}{x^2+9}$. (h) $\lim_{x\to\pi}\dfrac{\sin x}{x-\pi}$. (i) $\lim_{x\to 0}\dfrac{\sin x}{x+\sin x}$.

21.2 – Compute

$$\textbf{(i)}\ \lim_{x\to\infty}\frac{\sin x}{x}\qquad \textbf{(ii)}\ \lim_{x\to\infty}\frac{\cos x}{x}.$$

21.3 – Find a constant k such that the function

$$f(x)=\begin{cases}3x+2 & \text{for } x<2\\ x^2+k & \text{for } x\ge 2.\end{cases}$$

is continuous. Hint: Compute the one-sided limits.

21.4 – Find constants a and b such that the function

$$f(x)=\begin{cases}x^3 & \text{for } x<-1\\ ax+b & \text{for } -1\le x<1\\ x^2+2 & \text{for } x\ge 1.\end{cases}$$

is continuous for all x.

21.5 – Is there a constant k such that the function

$$f(x)=\begin{cases}\sin(1/x) & \text{for } x\ne 0\\ k & \text{for } x=0.\end{cases}$$

is continuous? If so, find it; if not, say why.

21.6 – Compute $\lim_{x\to\infty} x\sin\frac{\pi}{x}$ and $\lim_{x\to\infty} x\tan\frac{\pi}{x}$.

21.7 – Let A_n be the area of the regular $2n$-gon inscribed in the unit circle, and let B_n be the area of the regular $2n$-gon whose inscribed circle has radius 1.

Show that

$$A_n = 2^n \sin\frac{\pi}{2^n} \text{ and } B_n = 2^{n+1}\tan\frac{\pi}{2^{n+1}}$$

Compute $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} B_n$.

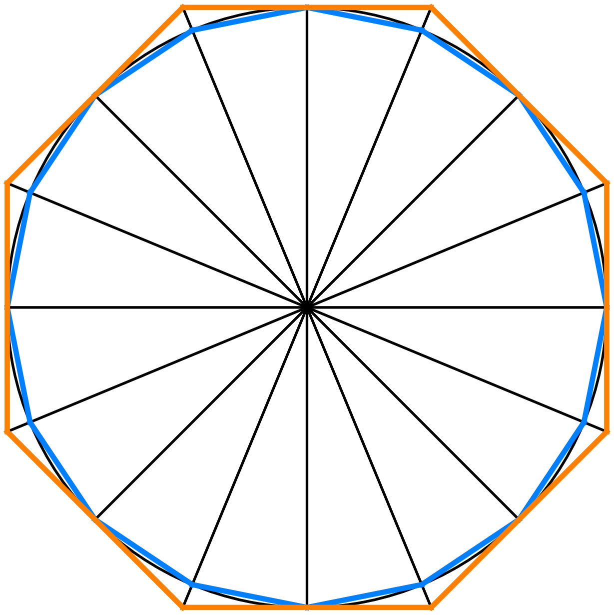


FIGURE 9. A_8, B_4 and π

IV. **Derivatives (2)**

> *“Leibniz never thought of the derivative as a limit. This does not appear until the work of d’Alembert.”*
>
> `http://www.gap-system.org/~history/Biographies/Leibniz.html`

In chapter II we saw two mathematical problems which led to expressions of the form $\frac{0}{0}$. Now that we know how to handle limits, we can state the definition of the derivative of a function. After computing a few derivatives using the definition we will spend most of this section developing the *differential calculus,* which is a collection of rules that allow you to compute derivatives without always having to use basic definition.

22. **Derivatives Defined**

Definition. 22.1. *Let f be a function which is defined on some interval (c, d) and let a be some number in this interval.*

The ***derivative of the function*** *f* ***at*** *a is the value of the limit*

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}. \tag{15}$$

f is said to be ***differentiable at*** *a if this limit exists.*

f is called ***differentiable on the interval*** *(c, d) if it is differentiable at every point a in (c, d).*

22.1. *Other notations*

One can substitute $x = a + h$ in the limit (15) and let $h \to 0$ instead of $x \to a$. This gives the formula

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}, \tag{16}$$

Often you will find this equation written with x instead of a and Δx instead of h, which makes it look like this:

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The interpretation is the same as in equation (8) from §10. The numerator $f(x + \Delta x) - f(x)$ represents the amount by which the function value of f changes if one increases its argument x by a (small) amount Δx. If you write $y = f(x)$ then we can call the increase in f

$$\Delta y = f(x + \Delta x) - f(x),$$

so that the derivative $f'(x)$ is

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}.$$

Gottfried Wilhelm von Leibniz, one of the inventors of calculus, came up with the idea that one should write this limit as

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x},$$

the idea being that after letting Δx go to zero it didn’t vanish, but instead became an infinitely small quantity which Leibniz called “dx.” The result of increasing x by this infinitely small quantity dx is that $y = f(x)$ increased by another infinitely small quantity

dy. The ratio of these two infinitely small quantities is what we call the derivative of $y = f(x)$.

There are no “infinitely small real numbers,” and this makes Leibniz’ notation difficult to justify. In the 20th century mathematicians have managed to create a consistent theory of “infinitesimals” which allows you to compute with “dx and dy” as Leibniz and his contemporaries would have done. This theory is called “non standard analysis.” We won’t mention it any further[3]. Nonetheless, even though we won’t use infinitely small numbers, Leibniz’ notation is very useful and we will use it.

23. **Direct computation of derivatives**

23.1. *Example – The derivative of* $f(x) = x^2$ *is* $f'(x) = 2x$

We have done this computation before in §8. The result was

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

Leibniz would have written

$$\frac{dx^2}{dx} = 2x.$$

23.2. *The derivative of* $g(x) = x$ *is* $g'(x) = 1$

Indeed, one has

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{(x+h) - x}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

In Leibniz’ notation:

$$\frac{dx}{dx} = 1.$$

This is an example where Leibniz’ notation is most misleading, because if you divide dx by dx then you should of course get 1. Nonetheless, this is not what is going on. The expression $\frac{dx}{dx}$ is not really a fraction since there are no two “infinitely small” quantities dx which we are dividing.

23.3. *The derivative of any constant function is zero*

Let $k(x) = c$ be a constant function. Then we have

$$k'(x) = \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

Leibniz would have said that if c is a constant, then

$$\frac{dc}{dx} = 0.$$

---

[3] But if you want to read more on this you should see Keisler’s calculus text at

`http://www.math.wisc.edu/~keisler/calc.html`

I would not recommend using Keisler’s text and this text at the same time, but if you like math you should remember that it exists, and look at it (later, say, after you pass 221.)

23.4. *Derivative of x^n for $n = 1, 2, 3, \ldots$*

To differentiate $f(x) = x^n$ one proceeds as follows:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^n - a^n}{x - a}.$$

We need to simplify the fraction $(x^n - a^n)/(x - a)$. For $n = 2$ we have

$$\frac{x^2 - a^2}{x - a} = x + a.$$

For $n = 1, 2, 3, \ldots$ the geometric sum formula tells us that

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}. \tag{17}$$

If you don't remember the geometric sum formula, then you could also just verify (17) by carefully multiplying both sides with $x - a$. For instance, when $n = 3$ you would get

$$\begin{array}{rclcccc} x \times (x^2 + xa + a^2) & = & x^3 & +ax^2 & +a^2x & \\ -a \times (x^2 + xa + a^2) & = & & -ax^2 & -a^2x & -a^3 \\ \hline (x - a) \times (x^2 + ax + a^2) & = & x^3 & & & -a^3 \end{array}$$

With formula (17) in hand we can now easily find the derivative of x^n:

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a} \left\{x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}\right\} \\ &= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots + a\,a^{n-2} + a^{n-1}. \end{aligned}$$

Here there are n terms, and they all are equal to a^{n-1}, so the final result is

$$f'(a) = na^{n-1}.$$

One could also write this as $f'(x) = nx^{n-1}$, or, in Leibniz' notation

$$\frac{dx^n}{dx} = nx^{n-1}.$$

This formula turns out to be true in general, but here we have only proved it for the case in which n is a positive integer.

23.5. *Differentiable implies Continuous*

Theorem 23.1. *If a function f is differentiable at some a in its domain, then f is also continuous at a.*

Proof. We are given that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists, and we must show that

$$\lim_{x\to a} f(x) = f(a).$$

This follows from the following computation

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}\big(f(x)-f(a)+f(a)\big) && \text{(algebra)}\\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)+f(a) && \text{(more algebra)}\\
&= \left\{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right\}\cdot\lim_{x\to a}(x-a)+\lim_{x\to a}f(a) && \text{(Limit Properties)}\\
&= f'(a)\cdot 0+f(a) && (f'(a)\text{ exists})\\
&= f(a).
\end{aligned}$$

□

23.6. *Some non-differentiable functions*

23.6.1. *A graph with a corner.* Consider the function

$$f(x)=|x|=\begin{cases} x & \text{for } x\geq 0,\\ -x & \text{for } x<0.\end{cases}$$

This function is continuous at all x, but it is not differentiable at $x=0$.

To see this try to compute the derivative at 0,

$$f'(0)=\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}=\lim_{x\to 0}\operatorname{sign}(x).$$

We know this limit does not exist (see §16.1)

If you look at the graph of $f(x)=|x|$ then you see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin.

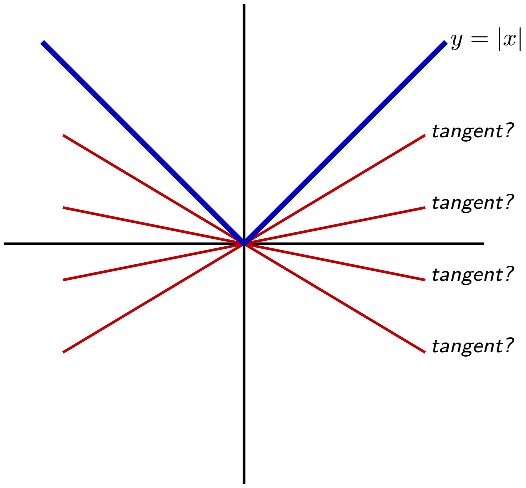


FIGURE 10. The graph of $y=|x|$ has no tangent at the origin.

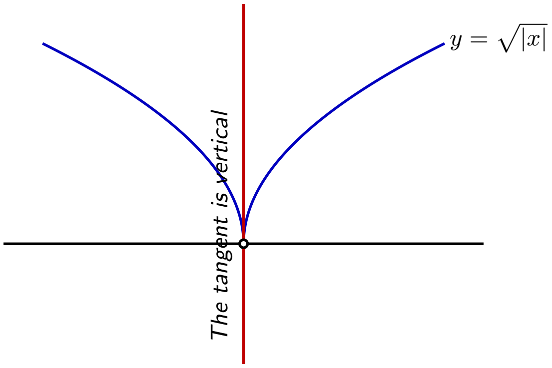


FIGURE 11. Tangent to the graph of $y = |x|^{1/2}$ at the origin

23.6.2. *A graph with a cusp.* Another example of a function without a derivative at $x = 0$ is

$$f(x) = \sqrt{|x|}.$$

When you try to compute the derivative you get this limit

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x|}}{x} = ?$$

The limit from the right is

$$\lim_{x\searrow 0} \frac{\sqrt{|x|}}{x} = \lim_{x\searrow 0} \frac{1}{\sqrt{x}},$$

which does not exist (it is "$+\infty$"). Likewise, the limit from the left also does not exist ('tis "$-\infty$). Nonetheless, a drawing for the graph of f suggests an obvious tangent to the graph at $x = 0$, namely, the y-axis. That observation does not give us a derivative, because the y-axis is vertical and hence has no slope.

23.6.3. *A graph with absolutely no tangents anywhere.* The previous two examples were about functions which did not have a derivative at $x = 0$. In both examples the point $x = 0$ was the only point where the function failed to have a derivative. It is easy to give examples of functions which are not differentiable at more than one value of x, but here I would like to show you a function f which doesn't have a derivative ***anywhere in its domain.***

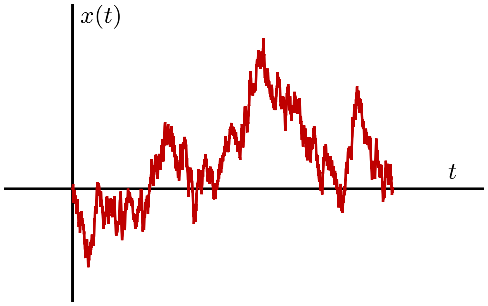


FIGURE 12. A one-dimensional Brownian motion

To keep things short I won't write a formula for the function, and merely show you a graph (see Figure 12.) In this graph you see a typical path of a Brownian motion, i.e. t is time, and $x(t)$ is the position of a particle which undergoes a Brownian motion – come to lecture for further explanation (see also the article on wikipedia).

To see a similar graph check the Dow Jones or Nasdaq in the upper left hand corner of the web page at `http://finance.yahoo.com` in the afternoon on any weekday.

Exercises

23.1 – Compute the derivative of the following functions

$$f(x) = x^2 - 2x \qquad g(x) = \frac{1}{x} \qquad k(x) = x^3 - 17x$$
$$u(x) = \frac{2}{1+x} \qquad v(x) = \sqrt{x} \qquad w(x) = \frac{1}{\sqrt{x}}$$

using either (15) or (16).

23.2 – Which of the following functions is differentiable at $x = 0$?

$$f(x) = x|x|, \qquad g(x) = x\sqrt{|x|}, \qquad h(x) = x + |x|.$$

23.3 – For which value(s) is the function defined by

$$f(x) = \begin{cases} ax + b & \text{for } x < 0 \\ x - x^2 & \text{for } x \ge 0 \end{cases}$$

differentiable at $x = 0$?

23.4 – For which value(s) is the function defined by

$$f(x) = \begin{cases} ax^2 & \text{for } x < 2 \\ x + b & \text{for } x \ge 2 \end{cases}$$

differentiable at $x = 0$?

23.5 – *True or false:* If a function f is continuous at some $x = a$ then it must also be differentiable at $x = a$?

23.6 – *True or false:* If a function f is differentiable at some $x = a$ then it must also be continuous at $x = a$?

24. The Differentiation Rules

You could go on and compute more derivatives from the definition. Each time you would have to compute a new limit, and hope that there is some trick that allows you to find that limit. This is fortunately not necessary. It turns out that if you know a few basic derivatives (such as $dx^n/dx = nx^{n-1}$) the you can find derivatives of arbitrarily complicated functions by breaking them into smaller pieces. In this section we'll look at rules which tell you how to differentiate a function which is either the sum, difference, product or quotient of two other functions.

The situation is analogous to that of the "limit-properties" $(P_1) \dots (P_6)$ from the previous chapter which allowed us to compute limits without always having to go back to the epsilon-delta definition.

24.1. *Sum, product and quotient rules*

In the following c and n are constants, u and v are functions of x, and $'$ denotes differentiation. The Differentiation Rules in function notation, and Leibniz notation, are

Constant rule:	$c' = 0$	$\dfrac{dc}{dx} = 0$
Sum rule:	$(u \pm v)' = u' \pm v'$	$\dfrac{du \pm v}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$
Product rule:	$(u \cdot v)' = u' \cdot v + u \cdot v'$	$\dfrac{duv}{dx} = \dfrac{du}{dx} v + u \dfrac{dv}{dx}$
Quotient rule:	$\left(\dfrac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}$	$\dfrac{d\dfrac{u}{v}}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Note that we already proved the Constant Rule in example 23.2. We will now prove the sum, product and quotient rules.

24.2. *Proof of the Sum Rule*

Suppose that $f(x) = u(x) + v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} && \text{(definition of } f'\text{)}\\
&= \lim_{x\to a} \frac{\big(u(x) + v(x)\big) - \big(u(a) + v(a)\big)}{x - a} && \text{(use } f = u + v\text{)}\\
&= \lim_{x\to a} \left(\frac{u(x) - u(a)}{x - a} + \frac{v(x) - v(a)}{x - a}\right) && \text{(algebra)}\\
&= \lim_{x\to a} \frac{u(x) - u(a)}{x - a} + \lim_{x\to a} \frac{v(x) - v(a)}{x - a} && \text{(limit property)}\\
&= u'(a) + v'(a) && \text{(definition of } u',\ v'\text{)}
\end{aligned}$$

24.3. *Proof of the Product Rule*

Let $f(x) = u(x)v(x)$. To find the derivative we must express the change of f in terms of the changes of u and v

$$\begin{aligned}
f(x) - f(a) &= u(x)v(x) - u(a)v(a)\\
&= u(x)v(x) - u(x)v(a) + u(x)v(a) - u(a)v(a)\\
&= u(x)\big(v(x) - v(a)\big) + \big(u(x) - u(a)\big)v(a)
\end{aligned}$$

Now divide by $x - a$ and let $x \to a$:

$$\begin{aligned}
\lim_{x\to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x\to a} u(x)\frac{v(x) - v(a)}{x - a} + \frac{u(x) - u(a)}{x - a} v(a)\\
&\qquad\qquad\text{(use the limit properties)}\\
&= \Big(\lim_{x\to a} u(x)\Big)\Big(\lim_{x\to a} \frac{v(x) - v(a)}{x - a}\Big) + \Big(\lim_{x\to a} \frac{u(x) - u(a)}{x - a}\Big)v(a)\\
&= u(a)v'(a) + u'(a)v(a),
\end{aligned}$$

as claimed. In this last step we have used that

$$\lim_{x\to a} \frac{u(x) - u(a)}{x - a} = u'(a) \quad \text{and} \quad \lim_{x\to a} \frac{v(x) - v(a)}{x - a} = v'(a)$$

and also that

$$\lim_{x\to a} u(x) = u(a)$$

This last limit follows from the fact that u is continuous, which in turn follows from the fact that u is differentiable.

24.4. *Proof of the Quotient Rule*

We can break the proof into two parts. First we do the special case where $f(x) = 1/v(x)$, and then we use the product rule to differentiate

$$f(x) = \frac{u(x)}{v(x)} = u(x) \cdot \frac{1}{v(x)}.$$

So let $f(x) = 1/v(x)$. We can express the change in f in terms of the change in v

$$f(x) - f(a) = \frac{1}{v(x)} - \frac{1}{v(a)} = \frac{v(x) - v(a)}{v(x)v(a)}.$$

Dividing by $x - a$ we get

$$\frac{f(x) - f(a)}{x - a} = \frac{1}{v(x)v(a)} \frac{v(x) - v(a)}{x - a}.$$

Now we want to take the limit $x \to a$. We are given the v is differentiable, so it must also be continuous and hence

$$\lim_{x\to a} v(x) = v(a).$$

Therefore we find

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{1}{v(x)v(a)} \lim_{x\to a} \frac{v(x) - v(a)}{x - a} = \frac{v'(a)}{v(a)^2}.$$

That completes the first step of the proof. In the second step we use the product rule to differentiate $f = u/v$

$$f' = \left(\frac{u}{v}\right)' = \left(u \cdot \frac{1}{v}\right)' = u' \cdot \frac{1}{v} + u \cdot \left(\frac{1}{v}\right)' = \frac{u'}{v} - u\frac{v'}{v^2} = \frac{u'v - uv'}{v^2}.$$

24.5. *A shorter, but not quite perfect derivation of the Quotient Rule*

The Quotient Rule can be derived from the Product Rule as follows: if $w = u/v$ then

$$w \cdot v = u \tag{18}$$

By the product rule we have

$$w' \cdot v + w \cdot v' = u',$$

so that

$$w' = \frac{u' - w \cdot v'}{v} = \frac{u' - (u/v) \cdot v'}{v} = \frac{u' \cdot v - u \cdot v'}{v^2}.$$

Unlike the proof in §24.4 above, this argument does not prove that w is differentiable if u and v are. It only says that ***if the derivative exists*** then it must be what the Quotient Rule says it is.

The trick which is used here, is a special case of a method called "implicit differentiation." We have an equation (18) which the quotient w satisfies, and from by differentiating this equation we find w'.

24.6. *Differentiating a constant multiple of a function*

Note that the rule

$$(cu)' = cu'$$

follows from the Constant Rule and the Product Rule.

24.7. *Picture of the Product Rule*

If u and v are quantities which depend on x, and if increasing x by Δx causes u and v to change by Δu and Δv, then the product of u and v will change by

$$\Delta(uv) = (u+\Delta u)(v+\Delta v) - uv = u\Delta v + v\Delta u + \Delta u\Delta v. \tag{19}$$

If u and v are differentiable functions of x, then the changes Δu and Δv will be of the same order of magnitude as Δx, and thus one expects $\Delta u\Delta v$ to be much smaller. One therefore ignores the last term in (19), and thus arrives at

$$\Delta(uv) = u\Delta v + v\Delta u.$$

Leibniz would now divide by Δx and replace Δ's by d's to get the product rule:

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x}.$$

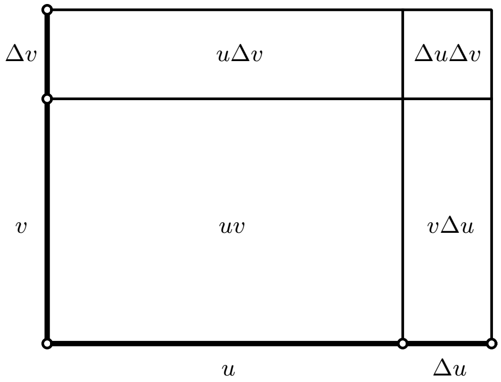


FIGURE 13. The Product Rule: how much does the area of a rectangle change if its sides u and v are increased by Δu and Δv?

25. **Differentiating powers of functions**

25.1. *Product rule with more than one factor*

If a function is given as the product of n functions, i.e.

$$f(x) = u_1(x) \times u_2(x) \times \cdots \times u_n(x),$$

then you can differentiate it by applying the product rule $n-1$ times (there are n factors, so there are $n-1$ multiplications.)

After the first step you would get

$$f' = u_1'\bigl(u_2\cdots u_n\bigr) + u_1\bigl(u_2\cdots u_n\bigr)'.$$

In the second step you apply the product rule to $(u_2u_3\cdots u_n)'$. This yields

$$\begin{aligned} f' &= u_1'u_2\cdots u_n + u_1\bigl[u_2'u_3\cdots u_n + u_2(u_3\cdots u_n)'\bigr] \\ &= u_1'u_2\cdots u_n + u_1u_2'u_3\cdots u_n + u_1u_2\bigl(u_3\cdots u_n\bigr)'. \end{aligned}$$

Continuing this way one finds after $n-1$ applications of the product rule that

$$\bigl(u_1\cdots u_n\bigr)' = u_1'u_2\cdots u_n + u_1u_2'u_3\cdots u_n + \cdots + u_1u_2u_3\cdots u_n'. \tag{20}$$

25.2. *The Power rule*

If all n factors in the previous paragraph are the same, so that the function f is the n^{th} power of some other function,

$$f(x) = \big(u(x)\big)^n,$$

then all terms in the right hand side of (20) are the same, and, since there are n of them, one gets

$$f'(x) = nu^{n-1}(x)u'(x),$$

or, in Leibniz' notation,

$$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}. \tag{21}$$

25.3. *The Power Rule for Negative Integer Exponents*

We have just proved the power rule (21) assuming n is a positive integer. The rule actually holds for all real exponents n, but the proof is harder.

Here we prove the Power Rule for negative exponents using the Quotient Rule. Suppose $n = -m$ where m is a positive integer. Then the Quotient Rule tells us that

$$(u^n)' = \big(u^{-m}\big)' = \left(\frac{1}{u^m}\right)' \overset{\text{Q.R.}}{=} -\frac{(u^m)'}{(u^m)^2}.$$

Since m is a positive integer, we can use (21), so $(u^m)' = mu^{m-1}$, and hence

$$(u^n)' = -\frac{mu^{m-1}\cdot u'}{u^{2m}} = -mu^{-m-1}\cdot u' = nu^{n-1}u'.$$

25.4. *The Power Rule for Rational Exponents*

So far we have proved that the power law holds if the exponent n is an integer.

We will now see how you can show that the power law holds even if the exponent n is any fraction, $n = p/q$. The following derivation contains the trick called ***implicit differentiation*** which we will study in more detail in chapter **??**.

So let $n = p/q$ where p and q are integers and consider the function

$$w(x) = u(x)^{p/q}.$$

Assuming that both u and w are differentiable functions, we will show that

$$w'(x) = \frac{p}{q}u(x)^{\frac{p}{q}-1}u'(x) \tag{22}$$

Raising both sides to the qth power gives

$$w(x)^q = u(x)^p.$$

Here the exponents p and q are integers, so we may apply the Power Rule to both sides. We get

$$qw^{q-1}\cdot w' = pu^{p-1}\cdot u'.$$

Dividing both sides by qw^{q-1} and substituting $u^{p/q}$ for w gives

$$w' = \frac{pu^{p-1}\cdot u'}{qw^{q-1}} = \frac{pu^{p-1}\cdot u'}{qu^{p(q-1)/q}} = \frac{pu^{p-1}\cdot u'}{qu^{p-(p/q)}} = \frac{p}{q}\cdot u^{(p/q)-1}\cdot u'$$

which is the Power Rule for $n = p/q$.

This proof is flawed because we did not show that $w(x) = u(x)^{p/q}$ is differentiable: we only showed what the derivative should be, *if it exists.*

25.5. *Derivative of x^n for integer n*

If you choose the function $u(x)$ in the Power Rule to be $u(x) = x$, then $u'(x) = 1$, and hence the derivative of $f(x) = u(x)^n = x^n$ is

$$f'(x) = nu(x)^{n-1}u'(x) = nx^{n-1} \cdot 1 = nx^{n-1}.$$

We already knew this of course.

25.6. *Example – differentiate a polynomial*

Using the Differentiation Rules you can easily differentiate any polynomial and hence any rational function. For example, using the Sum Rule, the Power Rule with $u(x) = x$, the rule $(cu)' = cu'$, the derivative of the polynomial

$$f(x) = 2x^4 - x^3 + 7$$

is

$$f'(x) = 8x^3 - 3x^2.$$

25.7. *Example – differentiate a rational function*

By the Quotient Rule the derivative of the function

$$g(x) = \frac{2x^4 - x^3 + 7}{1 + x^2}$$

is

$$\begin{aligned} g'(x) &= \frac{(8x^3 - 3x^2)(1 + x^2) - (2x^4 - x^3 + 7)2x}{(1 + x^2)^2} \\ &= \frac{6x^5 - x^4 + 8x^3 - 3x^2 - 14x}{(1 + x^2)^2}. \end{aligned}$$

If you compare this example with the previous then you see that polynomials simplify when you differentiate them while rational functions become more complicated.

25.8. *Derivative of the square root*

The derivative of $f(x) = \sqrt{x} = x^{1/2}$ is

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

where we used the power rule with $n = 1/2$ and $u(x) = x$.

Exercises

25.1 – Let $f(x) = (x^2 + 1)(x^3 + 3)$. Find $f'(x)$ in two ways:

(**i**) by multiplying and then differentiating,

(**ii**) by using the product rule.

Are your answers the same?

25.2 – Let $f(x) = (1+x^2)^4$. Find $f'(x)$ in two ways, first by expanding to get an expression for $f(x)$ as a polynomial in x and then differentiating, and then by using the power rule. Are the answers the same?

25.3 – Prove the statement in §24.6, i.e. show that $(cu)' = c(u')$ follows from the product rule.

25.4 – Compute the derivatives of the following functions (try to simplify your answers)

(i) $f(x) = x + 1 + (x + 1)^2$ **(ii)** $f(x) = \dfrac{x - 2}{x^4 + 1}$ **(iii)** $f(x) = \big(\dfrac{1}{1 + x}\big)^{-1}$

(iv) $f(x) = \sqrt{1 - x^2}$ **(v)** $f(x) = \dfrac{ax + b}{cx + d}$ **(vi)** $f(x) = \dfrac{1}{(1 + x^2)^2}$

(vii) $f(x) = \dfrac{x}{1 + \sqrt{x}}$ **(viii)** $f(x) = \sqrt{\dfrac{1 - x}{1 + x}}$ **(ix)** $f(x) = \sqrt[3]{x + \sqrt{x}}$

(x) $\varphi(t) = \dfrac{t}{1 + \sqrt{t}}$ **(xi)** $g(s) = \sqrt{\dfrac{1 - s}{1 + s}}$ **(xii)** $h(\rho) = \sqrt[3]{\rho + \sqrt{\rho}}$

25.5 *Using derivatives to approximate numbers.*

(i) Find the derivative of $f(x) = x^{4/3}$.

(ii) Use **(i)** to estimate the number

$$\frac{127^{4/3} - 125^{4/3}}{2}$$

approximately without a calculator. Your answer should have the form p/q where p and q are integers. [*Hint: Note that* $5^3 = 125$ *and take a good look at equation* (15).]

(iii) Approximate in the same way the numbers $\sqrt{143}$ and $\sqrt{145}$ (Hint: $12 \times 12 = 144$).

25.6 *Making the product and quotient rules look nicer.* Instead of looking at the derivative of a function you can look at the ratio of its derivative to the function itself, i.e. you can compute f'/f. This quantity is called the ***logarithmic derivative of the function*** f for reasons that will become clear later this semester.

(i) Compute the logarithmic derivative of these functions (i.e. find $f'(x)/f(x)$)

$$f(x) = x \qquad g(x) = 3x \qquad h(x) = x^2$$
$$k(x) = -x^2 \qquad \ell(x) = 2007x^2 \qquad m(x) = x^{2007}$$

(ii) Show that for any pair of functions u and v one has

$$\frac{(uv)'}{uv} = \frac{u'}{u} + \frac{v'}{v}$$
$$\frac{(u/v)'}{u/v} = \frac{u'}{u} - \frac{v'}{v}$$
$$\frac{\big(u^n\big)'}{u^n} = n\,\frac{u'}{u}$$

25.7 – **(i)** Find $f'(x)$ and $g'(x)$ if

$$f(x) = \frac{1 + x^2}{2x^4 + 7}, \qquad g(x) = \frac{2x^4 + 7}{1 + x^2}.$$

Note that $f(x) = 1/g(x)$.

(ii) Is it true that $f'(x) = 1/g'(x)$?

(iii) Is it true that $f(x) = g^{-1}(x)$?

(iv) Is it true that $f(x) = g(x)^{-1}$?

25.8 – **(i)** Let $x(t) = (1 - t^2)/(1 + t^2)$, $y(t) = 2t/(1 + t^2)$ and $u(t) = y(t)/x(t)$. Find dx/dt, dy/dt.

(ii) Now that you've done **(i)** there are two different ways of finding du/dt. What are they, and use one of both to find du/dt.

26. **Higher Derivatives**

26.1. *The derivative is a function*

If the derivative $f'(a)$ of some function f exists for all a in the domain of f, then we have a new function: namely, for each number in the domain of f we compute the derivative of f at that number. This function is called the ***derivative function*** of f, and it is denoted by f'. Now that we have agreed that the derivative of a function is a function, we can repeat the process and try to differentiate the derivative. The result, if it exists, is called the ***second derivative of*** f. It is denoted f''. The derivative of the second derivative is called the third derivative, written f''', and so on.

The nth derivative of f is denoted $f^{(n)}$. Thus

$$f^{(0)} = f, \qquad f^{(1)} = f', \qquad f^{(2)} = f'', \qquad f^{(3)} = f''', \ldots .$$

Leibniz' notation for the nth derivative of $y = f(x)$ is

$$\frac{d^n y}{dx^n} = f^{(n)}(x).$$

26.2. *Operator notation*

A common variation on Leibniz' notation for derivatives is the so-called ***operator notation***, as in

$$\frac{d(x^3 - x)}{dx} = \frac{d}{dx}(x^3 - x) = 3x^2 - 1.$$

For higher derivatives one can write

$$\frac{d^2 y}{dx^2} = \left(\frac{d}{dx}\right)^2 y$$

Be careful to distinguish the second derivative from the square of the first derivative. Usually

$$\frac{d^2 y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2 \textbf{!!!!}$$

Exercises

26.1 – The equation

$$(\dagger) \qquad \frac{2x}{x^2 - 1} = \frac{1}{x+1} + \frac{1}{x-1}$$

holds for all values of x (except $x = \pm 1$), so you should get the same answer if you differentiate both sides. Check this.

Compute the third derivative of $f(x) = 2x/(x^2 - 1)$ by using either the left or right hand side (your choice) of (†).

26.2 – Compute the first, second and third derivatives of the following functions

$$f(x) = (x+1)^4 \qquad g(x) = \left(x^2 + 1\right)^4$$

$$h(x) = \sqrt{x - 2} \qquad k(x) = \sqrt[3]{x - \frac{1}{x}}$$

26.3 – Find the derivatives of 10^{th} order of the functions

$$f(x) = x^{12} + x^8 \qquad g(x) = \frac{1}{x}$$
$$h(x) = \frac{12}{1-x} \qquad k(x) = \frac{x^2}{1-x}$$

26.4 – Find $f'(x)$, $f''(x)$ and $f^{(3)}(x)$ if

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}.$$

26.5 – (**i**) Find the 12^{th} derivative of the function $f(x) = \dfrac{1}{x+2}$.

(**ii**) Find the n^{th} order derivative of $f(x) = \dfrac{1}{x+2}$(i.e. find a formula for $f^{(n)}(x)$ which is valid for all $n = 0, 1, 2, 3\ldots$).

(**iii**) Find the n^{th} order derivative of $g(x) = \dfrac{x}{x+2}$.

26.6 *About notation*

(**i**) Find dy/dx and d^2y/dx^2 if $y = x/(x+2)$. Hint: See previous problem.

(**ii**) Find du/dt and d^2u/dt^2 if $u = t/(t+2)$. Hint: See previous problem.

(**iii**) Find $\dfrac{d}{dx}\left(\dfrac{x}{x+2}\right)$ and $\dfrac{d^2}{dx^2}\left(\dfrac{x}{x+2}\right)$. Hint: See previous problem.

(**iv**) Find $\dfrac{d}{dx}\left(\dfrac{x}{x+2}\right)\Bigg|_{x=1}$ and $\dfrac{d}{dx}\left(\dfrac{1}{1+2}\right)$.

26.7 – Find d^2y/dx^2 and $(dy/dx)^2$ if $y = x^3$.

27. Differentiating Trigonometric functions

The trigonometric functions Sine, Cosine and Tangent are differentiable, and their derivatives are given by the following formulas

$$\frac{d\sin x}{dx} = \cos x, \quad \frac{d\cos x}{dx} = -\sin x, \quad \frac{d\tan x}{dx} = \frac{1}{\cos^2 x}. \tag{23}$$

Note the minus sign in the derivative of the cosine!

Proof. By definition one has

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}.$$

To simplify the numerator we use the trigonometric addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

with $\alpha = x$ and $\beta = h$, which results in

$$\begin{aligned}\frac{\sin(x+h) - \sin(x)}{h} &= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \cos(x)\frac{\sin(h)}{h} + \sin(x)\frac{\cos(h)-1}{h}\end{aligned}$$

Hence by the formulas

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$$

from Section 21 we have

$$\begin{aligned}\sin'(x) &= \lim_{h\to 0} \cos(x)\frac{\sin(h)}{h} + \sin(x)\frac{\cos(h)-1}{h}\\ &= \cos(x)\cdot 1 + \sin(x)\cdot 0\\ &= \cos(x).\end{aligned}$$

A similar computation leads to the stated derivative of $\cos x$.

To find the derivative of $\tan x$ we apply the quotient rule to

$$\tan x = \frac{\sin x}{\cos x} = \frac{f(x)}{g(x)}.$$

We get

$$\tan'(x) = \frac{\cos(x)\sin'(x) - \sin(x)\cos'(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

as claimed. □

Exercises

27.1 – Find the derivatives of the following functions (try to simplify your answers)

(i) $f(x) = \sin(x) + \cos(x)$
(ii) $f(x) = 2\sin(x) - 3\cos(x)$
(iii) $f(x) = 3\sin(x) + 2\cos(x)$
(iv) $f(x) = x\sin(x) + \cos(x)$
(v) $f(x) = x\cos(x) - \sin x$
(vi) $f(x) = \dfrac{\sin x}{x}$
(vii) $f(x) = \cos^2(x)$
(viii) $f(x) = \sqrt{1 - \sin^2 x}$
(ix) $f(x) = \sqrt{\dfrac{1 - \sin x}{1 + \sin x}}$
(x) $\cot(x) = \dfrac{\cos x}{\sin x}$.

27.2 – Can you find a and b so that the function

$$f(x) = \begin{cases} \cos x & \text{for } x \le \frac{\pi}{4} \\ a + bx & \text{for } x > \frac{\pi}{4} \end{cases}$$

is differentiable at $x = \pi/4$?

27.3 – Can you find a and b so that the function

$$f(x) = \begin{cases} \tan x & \text{for } x < \frac{\pi}{6} \\ a + bx & \text{for } x \ge \frac{\pi}{6} \end{cases}$$

is differentiable at $x = \pi/4$?

27.4 – If f is a given function, and you have another function g which satisfies $g(x) = f(x) + 12$ for all x, then f and g have the same derivatives. Prove this. [*Hint: it's a short proof – use the differentiation rules.*]

27.5 – Show that the functions

$$f(x) = \sin^2 x \text{ and } g(x) = -\cos^2 x$$

have the same derivative by computing $f'(x)$ and $g'(x)$.

With hindsight this was to be expected – why?

28. **The Chain Rule**

28.1. *Composition of functions*

Given two functions f and g, one can define a new function called the ***composition of*** f ***and*** g. The notation for the composition is $f \circ g$, and it is defined by the formula

$$f \circ g(x) = f\big(g(x)\big).$$

The domain of the composition is the set of all numbers x for which this formula gives you something well-defined.

For instance, if $f(x) = x^2 + x$ and $g(x) = 2x + 1$ then

$$\begin{aligned} f \circ g(x) &= f\big(2x+1\big) = (2x+1)^2 + (2x+1) \\ \text{and } g \circ f(x) &= g\big(x^2+x\big) = 2(x^2+x) + 1 \end{aligned}$$

Note that $f \circ g$ and $g \circ f$ are not the same fucntion in this example (they hardly ever are the same).

If you think of functions as expressing dependence of one quantity on another, then the composition of functions arises as follows. If a quantity z is a function of another quantity y, and if y itself depends on x, then z depends on x via y.

To get $f \circ g$ from the previous example, we could say $z = f(y)$ and $y = g(x)$, so that

$$z = f(y) = y^2 + y \text{ and } y = 2x + 1.$$

Give x one can compute y, and from y one can then compute z. The result will be

$$z = y^2 + y = (2x+1)^2 + (2x+1),$$

in other notation,

$$z = f(y) = f\big(g(x)\big) = f \circ g(x).$$

One says that ***the composition of*** f ***and*** g ***is the result of subsituting*** g ***in*** f.

28.2. *A real world example*

A biologist is studying growth of yeast-cells. Assuming that a yeast-cell is spherical one can say how large it is by specifying its radius R. For a growing cell this radius will change with time t. The volume of the cell is a function of its radius, since the volume of a sphere of radius r is given by

$$V = \frac{4}{3}\pi r^3.$$

We now have two functions, the first f turns tells you the radius R of the cell at time t,

$$R = f(t)$$

and the second tells you the volume of the cell given its radius

$$V = g(r).$$

The volume of the cell at time t is then given by

$$V = g\big(f(t)\big) = g \circ f(t),$$

i.e. the function which tells you the volume of the cell at time t is the composition of first f and then g.

28.3. *Statement of the Chain Rule*

The chain rule tells you how to find the derivative of the composition $f \circ g$ of two functions f and g provided you now how to differentiate the two functions f and g.

Theorem 28.1 (*Chain Rule*). *If f and g are differentiable, so is the composition $f \circ g$*

The derivative of $f \circ g$ is given by

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

When written in Leibniz' notation the chain rule looks particularly easy. Suppose that $y = g(x)$ and $z = f(y)$, then $z = f \circ g(x)$, and the derivative of z with respect to x is the derivative of the function $f \circ g$. The derivative of z with respect to y is the derivative of the function f, and the derivative of y with respect to x is the derivative of the function g. In short,

$$\frac{dz}{dx} = (f \circ g)'(x), \quad \frac{dz}{dy} = f'(y) \quad \text{and} \quad \frac{dy}{dx} = g'(x)$$

so that the chain rule says

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx}. \tag{24}$$

First proof of the chain rule (using Leibniz' notation). We first consider difference quotients instead of derivatives, i.e. using the same notation as above, we consider the effect of an increase of x by an amount Δx on the quantity z.

If x increases by Δx, then $y = g(x)$ will increase by

$$\Delta y = g(x + \Delta x) - g(x),$$

and $z = f(y)$ will increase by

$$\Delta z = f(y + \Delta y) - f(y).$$

The ratio of the increase in $z = f(g(x))$ to the increase in x is

$$\frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta y} \cdot \frac{\Delta y}{\Delta x}.$$

In contrast to dx, dy and dz in equation (24), the Δx, etc. here are finite quantities, so this equation is just algebra: you can cancel the two Δys. If you let the increase Δx go to zero, then the increase Δy will also go to zero, and the difference quotients converge to the derivatives,

$$\frac{\Delta z}{\Delta x} \longrightarrow \frac{dz}{dx}, \quad \frac{\Delta z}{\Delta y} \longrightarrow \frac{dz}{dy}, \quad \frac{\Delta y}{\Delta x} \longrightarrow \frac{dy}{dx}$$

which immediately leads to Leibniz' form of the quotient rule. □

Proof of the chain rule. We verify the formula in Theorem 28.1 at some arbitrary value $x = a$, i.e. we will show that

$$(f \circ g)'(a) = f'(g(a))\, g'(a).$$

By definition the left hand side is

$$(f \circ g)'(a) = \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}.$$

The two derivatives on the right hand side are given by

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

and

$$f'(g(a)) = \lim_{y \to a} \frac{f(y) - f(g(a))}{y - g(a)}.$$

Since g is a differentiable function it must also be a continuous function, and hence $\lim_{x\to a} g(x) = g(a)$. So we can substitute $y = g(x)$ in the limit defining $f'(g(a))$

$$f'(g(a)) = \lim_{y\to a} \frac{f(y) - f(g(a))}{y - g(a)} = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}. \tag{25}$$

Put all this together and you get

$$\begin{aligned}(f \circ g)'(a) &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} \\ &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\ &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \\ &= f'(g(a)) \ \cdot \ g'(a)\end{aligned}$$

which is what we were supposed to prove – the proof seems complete.

There is one flaw in this proof, namely, we have divided by $g(x) - g(a)$, which is not allowed when $g(x) - g(a) = 0$. This flaw can be fixed but we will not go into the details here.[4] □

28.4. *First example*

We go back to the functions

$$z = f(y) = y^2 + y \text{ and } y = g(x) = 2x + 1$$

from the beginning of this section. The composition of these two functions is

$$z = f(g(x)) = (2x + 1)^2 + (2x + 1) = 4x^2 + 6x + 2.$$

We can compute the derivative of this composed function, i.e. the derivative of z with respect to x in two ways. First, you simply differentiate the last formula we have:

$$\frac{dz}{dx} = \frac{d(4x^2 + 6x + 2)}{dx} = 8x + 6. \tag{26}$$

The other approach is to use the chain rule:

$$\frac{dz}{dy} = \frac{d(y^2 + y)}{dy} = 2y + 1,$$

and

$$\frac{dy}{dx} = \frac{d(2x + 1)}{dx} = 2.$$

Hence, by the chain rule one has

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = (2y + 1) \cdot 2 = 4y + 2. \tag{27}$$

The two answers (26) and (27) should be the same. Once you remember that $y = 2x + 1$ you see that this is indeed true:

$$y = 2x + 1 \implies 4y + 2 = 4(2x + 1) + 2 = 8x + 6.$$

The two computations of dz/dx therefore lead to the same answer. In this example there was no clear advantage in using the chain rule. The chain rule becomes useful when the functions f and g become more complicated.

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[4] Briefly, you have to show that the function

$$h(y) = \begin{cases} \{f(y) - f(g(a))\}/(y - g(a)) & y \neq a \\ f'(g(a)) & y = a \end{cases}$$

is continuous.

28.5. *Example where you really need the Chain Rule*

We know what the derivative of $\sin x$ with respect to x is, but none of the rules we have found so far tell us how to differentiate $f(x) = \sin(2x)$.

The function $f(x) = \sin 2x$ is the composition of two simpler functions, namely

$$f(x) = g(h(x)) \text{ where } g(u) = \sin u \text{ and } h(x) = 2x.$$

We know how to differentiate each of the two functions g and h:

$$g'(u) = \cos u, \quad h'(x) = 2.$$

Therefore the chain rule implies that

$$f'(x) = g'(h(x))h'(x) = \cos(2x) \cdot 2 = 2\cos 2x.$$

Leibniz would have decomposed the relation $y = \sin 2x$ between y and x as

$$y = \sin u, \quad u = 2x$$

and then computed the derivative of $\sin 2x$ with respect to x as follows

$$\frac{d\sin 2x}{dx} \overset{u=2x}{=} \frac{d\sin u}{dx} = \frac{d\sin u}{du} \cdot \frac{du}{dx} = \cos u \cdot 2 = 2\cos 2x.$$

28.6. *The Power Rule and the Chain Rule*

The Power Rule, which says that for any function f and any rational number n one has

$$\frac{d}{dx}\bigl(f(x)^n\bigr) = nf(x)^{n-1}f'(x),$$

is a special case of the Chain Rule, for one can regard $y = f(x)^n$ as the composition of two functions

$$y = g(u), \quad u = f(x)$$

where $g(u) = u^n$. Since $g'(u) = nu^{n-1}$ the Chain Rule implies that

$$\frac{du^n}{dx} = \frac{du^n}{du} \cdot \frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

Setting $u = f(x)$ and $\frac{du}{dx} = f'(x)$ then gives you the Power Rule.

28.7. *The volume of a growing yeast cell*

Consider the "real world example" from §28.2 again. There we considered a growing spherical yeast cell of radius

$$r = f(t).$$

The volume of this cell is

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi f(t)^3.$$

We can regard this as the composition of two functions, $V = g(r) = \frac{4}{3}\pi r^3$ and $r = f(t)$.

According to the chain rule the rate of change of the volume with time is now

$$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt}$$

i.e. it is the product of the rate of change of the volume with the radius of the cell and the rate of change of the cell's radius with time. From

$$\frac{dV}{dr} = \frac{d\frac{4}{3}\pi r^3}{dr} = 4\pi r^2$$

we see that

$$\frac{dV}{dr} = 4\pi r^2\,\frac{dr}{dt}.$$

For instance, if the radius of the cell is growing at 0.5μm/sec, and if its radius is $r = 3.0\mu$m, then the volume is growing at a rate of

$$\frac{dV}{dt} = 4\pi(3.0\mu\text{m})^2 \times 0.5\mu\text{m/sec} \approx 57\mu\text{m}^3/\text{sec}.$$

28.8. *A more complicated example*

Suppose you needed to find the derivative of

$$y = h(x) = \frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2}$$

We can write this function as a composition of two simpler functions, namely,

$$y = f(u), \quad u = g(x),$$

with

$$f(u) = \frac{u}{(u+1)^2} \text{ and } g(x) = \sqrt{x+1}.$$

The derivatives of f and g are

$$f'(u) = \frac{1\cdot(u+1)^2 - u\cdot 2(u+1)}{(u+1)^4} = \frac{u+1-2}{(u+1)^3} = \frac{u-1}{(u+1)^3},$$

and

$$g'(x) = \frac{1}{2\sqrt{x+1}}.$$

Hence the derivative of the composition is

$$h'(x) = \frac{d}{dx}\left\{\frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2}\right\} = f'(u)g'(x) = \frac{u-1}{(u+1)^3} \cdot \frac{1}{2\sqrt{x+1}}.$$

The result should be a function of x, and we achieve this by replacing all u's with $u = \sqrt{x+1}$:

$$\frac{d}{dx}\left\{\frac{\sqrt{x+1}}{(\sqrt{x+1}+1)^2}\right\} = \frac{\sqrt{x+1}-1}{(\sqrt{x+1}+1)^3} \cdot \frac{1}{2\sqrt{x+1}}.$$

The last step (where you replace u by its definition in terms of x) is important because the problem was presented to you with only x and y as variables while u was a variable you introduced yourself to do the problem.

Sometimes it is possible to apply the Chain Rule without introducing new letters, and you will simply think "the derivative is the derivative of the outside with respect to the inside times the derivative of the inside." For instance, to compute

$$\frac{d\, 4+\sqrt{7+x^3}}{dx}$$

you could set $u = 7 + x^3$, and compute

$$\frac{d\, 4+\sqrt{7+x^3}}{dx} = \frac{d\, 4+\sqrt{u}}{du} \cdot \frac{du}{dx}.$$

Instead of writing all this explicitly, you could think of $u = 7+x^3$ as the function "inside the square root," and think of $4+\sqrt{u}$ as "the outside function." You would then immediately write

$$\frac{d}{dx}(4+\sqrt{7+x^3}) = \frac{1}{2\sqrt{7+x^3}} \cdot 3x^2.$$

28.9. *The Chain Rule and composing more than two functions*

Often we have to apply the Chain Rule more than once to compute a derivative. Thus if $y = f(u)$, $u = g(v)$, and $v = h(x)$ we have

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

In functional notation this is

$$(f \circ g \circ h)'(x) = f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x).$$

Note that each of the three derivatives on the right is evaluated at a different point. Thus if $b = h(a)$ and $c = g(b)$ the Chain Rule is

$$\left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{dy}{du}\right|_{u=c} \cdot \left.\frac{du}{dv}\right|_{v=b} \cdot \left.\frac{dv}{dx}\right|_{x=a}.$$

For example, if $y = \dfrac{1}{1+\sqrt{9+x^2}}$, then $y = 1/(1+u)$ where $u = 1+\sqrt{v}$ and $v = 9+x^2$ so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = -\frac{1}{(1+u)^2} \cdot \frac{1}{2\sqrt{v}} \cdot 2x.$$

so

$$\left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{dy}{du}\right|_{u=6} \cdot \left.\frac{du}{dv}\right|_{v=25} \cdot \left.\frac{dv}{dx}\right|_{x=4} = -\frac{1}{7} \cdot \frac{1}{10} \cdot 8.$$

Exercises

28.1 – Let $y = \sqrt{1+x^3}$ and find dy/dx using the Chain Rule. Say what plays the role of $y = f(u)$ and $u = g(x)$.

28.2 – Repeat the previous exercise with $y = (1+\sqrt{1+x})^3$.

28.3 – Alice and Bob differentiated $y = \sqrt{1+x^3}$ with respect to x differently. Alice wrote $y = \sqrt{u}$ and $u = 1+x^3$ while Bob wrote $y = \sqrt{1+v}$ and $v = x^3$. Assuming neither one made a mistake, did they get the same answer?

28.4 – Let $y = u^3+1$ and $u = 3x+7$. Find $\dfrac{dy}{dx}$ and $\dfrac{dy}{du}$. Express the former in terms of x and the latter in terms of u.

28.5 – Suppose that $f(x) = \sqrt{x}$, $g(x) = 1+x^2$, $v(x) = f \circ g(x)$, $w(x) = g \circ f(x)$. Find formulas for $v(x)$, $w(x)$, $v'(x)$, and $w'(x)$.

28.6 – Compute the following derivatives

(i) $f(x) = \sin 2x - \cos 3x$ **(ii)** $f(x) = \sin \dfrac{\pi}{x}$

(iii) $f(x) = \sin(\cos 3x)$ **(iv)** $f(x) = \dfrac{\sin x^2}{x^2}$

(v) $f(x) = \tan\sqrt{1+x^2}$ **(vi)** $f(x) = \cos^2 x - \cos x^2$.

28.7 – Suppose that $f(x) = x^2+1$, $g(x) = x+5$, and

$$v = f \circ g, \qquad w = g \circ f, \qquad p = f \cdot g, \qquad q = g \cdot f.$$

Find $v(x)$, $w(x)$, $p(x)$, and $q(x)$.

28.8 – Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	-4	-1/3	-8/3

Define

$$v(x) = f(g(x)), \qquad w(x) = g(f(x)), \qquad p(x) = f(x)g(x), \qquad q(x) = g(x)f(x).$$

Evaluate $v(0)$, $w(0)$, $p(0)$, $q(0)$, $v'(0)$ and $w'(0)$, $p'(0)$, $q'(0)$. If there is insufficient information to answer the question, so indicate.

28.9 – A differentiable function f satisfies $f(3) = 5$, $f(9) = 7$, $f'(3) = 11$ and $f'(9) = 13$. Find an equation for the tangent line to the curve $y = f(x^2)$ at the point $(x, y) = (3, 7)$.

28.10 – There is a function f whose second derivative satisfies

$$(\dagger) \qquad f''(x) = -64f(x).$$

(i) One such function is $f(x) = \sin ax$, provided you choose the right constant a: Which value should a have?

(ii) Can you find other functions f that satisfy (†)

28.11 – A cubical sponge is absorbing water, which causes it to expand. Its side at time t is $S(t)$. Its volume is $V(t)$.

(i) What is the relation between $S(t)$ and $V(t)$ (i.e. can you find a function f so that $V(t) = f(S(t))$)?

(ii) Describe the meaning of the derivatives $S'(t)$ and $V'(t)$ in one plain english sentence each. If we measure lengths in inches and time in minutes, then what units do $t, S(t), V(t), S'(t)$ and $V'(t)$ have?

(iii) What is the relation between $S'(t)$ and $V'(t)$?

(iv) At the moment that the sponge's volume is 8 cubic inches, it is absorbing water at a rate of 2 cubic inch per minute. How fast is its side $S(t)$ growing?