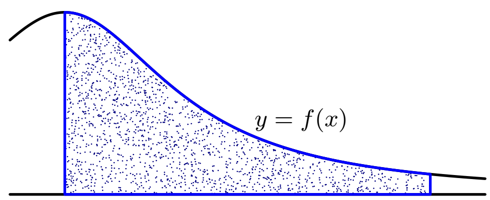
VII. **The Integral**

In this chapter we define the integral of a function on some interval $[a, b]$. The most common interpretation of the integral is in terms of the area under the graph of the given function, so that is where we begin.

50. **Area under a Graph**



Let f be a function which is defined on some interval $a \le x \le b$ and assume it is positive, i.e. assume that its graph lies above the x axis. *How large is the area of the region caught between the x axis, the graph of $y = f(x)$ and the vertical lines $y = a$ and $y = b$?*

One can try to compute this area by approximating the region with many thin rectangles. To do this you choose a *partition* of the interval $[a, b]$, i.e. you pick numbers $x_1 < \cdots < x_n$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

These numbers split the interval $[a, b]$ into n sub-intervals

$$[x_0, x_1], \quad [x_1, x_2], \quad \ldots, \quad [x_{n-1}, x_n]$$

whose lengths are

$$\Delta x_1 = x_1 - x_0, \quad \Delta x_2 = x_2 - x_1, \quad \ldots, \quad \Delta x_n = x_n - x_{n-1}.$$

In each interval we choose a point c_k, i.e. in the first interval we choose $x_0 \le c_1 \le x_1$, in the second interval we choose $x_1 \le c_2 \le x_2$, ..., and in the last interval we choose some number $x_{n-1} \le c_n \le x_n$. See figure 29.

We then define n rectangles: the base of the k^{th} rectangle is the interval $[x_{k-1}, x_k]$ on the x-axis, while its height is $f(c_k)$ (here k can be any integer from 1 to n.)

The area of the k^{th} rectangle is of course the product of its height and width, i.e. its area is $f(c_k)\Delta x_k$. Adding these we see that the total area of the rectangles is

$$R = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n. \tag{44}$$

This kind of sum is called a ***Riemann sum.***

If the partition is sufficiently fine then one would expect this sum, i.e. the total area of all rectangles to be a good approximation of the area of the region under the graph. Replacing the partition by a finer partition, with more division points, should improve the approximation. So you would expect the area to be the limit of Riemann-sums like R "as the partition becomes finer and finer." A precise formulation of the definition goes like this:

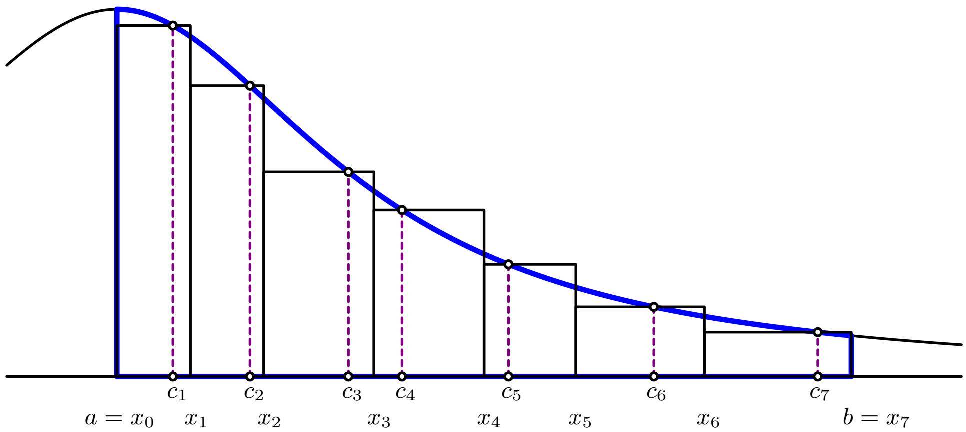


FIGURE 29. Forming a Riemann sum

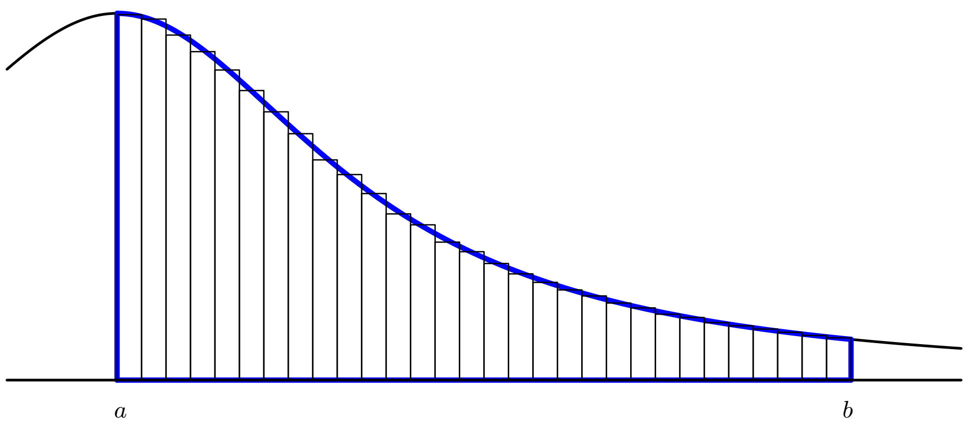


FIGURE 30. Refining the partition.

Definition. 50.1. *If f is a function defined on an interval $[a, b]$, then we say that*

$$\int_a^b f(x)dx = I,$$

i.e. the integral of "$f(x)$ from $x = a$ to b" equals I, if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$\Big| f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n - I \Big| < \varepsilon$$

holds for every partition all of whose intervals have length $\Delta x_k < \delta$.

51. When f changes its sign

If the function f is not necessarily positive everywhere in the interval $a \le x \le b$, then we still define the integral in exactly the same way: as a limit of Riemann sums whose mesh size becomes smaller and smaller. However the interpretation of the integral as "the area of the region between the graph and the x-axis" has a twist to it.

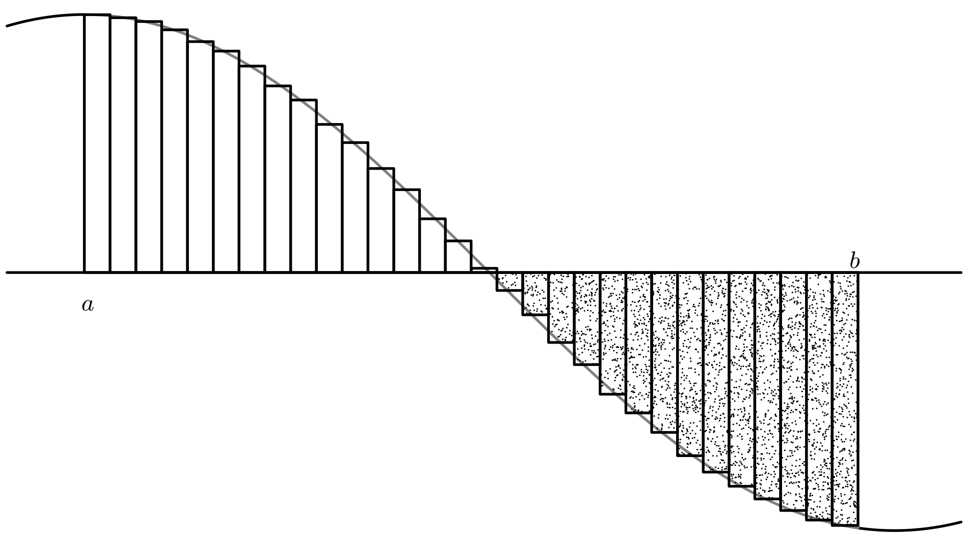


FIGURE 31. Illustrating a Riemann sum for a function whose sign changes

Let f be some function on an interval $a \le x \le b$, and form the Riemann sum

$$R = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n$$

that goes with some partition, and some choice of c_k.

When f can be positive or negative, then the terms in the Riemann sum can also be positive or negative. If $f(c_k) > 0$ then the quantity $f(c_k)\Delta x_k$ is the area of the corresponding rectangle, but if $f(c_k) < 0$ then $f(c_k)\Delta x_k$ is a negative number, namely *minus* the area of the corresponding rectangle. The Riemann sum is therefore the area of the rectangles above the x-axis *minus* the area below the axis and above the graph.

Taking the limit over finer and finer partitions, we conclude that

$$\int_a^b f(x)dx = \begin{array}{c}\text{area above the } x\text{-axis, below the graph} \\ \textbf{\textit{minus}} \text{ the area below the } x\text{-axis, above the graph.}\end{array}$$

52. The Fundamental Theorem of Calculus

Definition. 52.1. *A function F is called an **antiderivative** of f on the interval $[a, b]$ if one has $F'(x) = f(x)$ for all x with $a < x < b$.*

For instance, $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$, but so is $G(x) = \frac{1}{2}x^2 + 2008$.

Theorem 52.2. *If f is a function whose integral $\int_a^b f(x)dx$ exists, and if F is an antiderivative of f on the interval $[a,b]$, then one has*

$$\int_a^b f(x)dx = F(b) - F(a). \tag{45}$$

(a proof was given in lecture.)

Because of this theorem the expression on the right appears so often that various abbreviations have been invented. We will abbreviate

$$F(b) - F(a) \stackrel{\text{def}}{=} \big[F(x)\big]_{x=a}^{b} = \big[F(x)\big]_{a}^{b}.$$

52.1. *Terminology*

In the integral

$$\int_a^b f(x)\, dx$$

the numbers a and b are called the *bounds of the integral*, the function $f(x)$ which is being integrated is called *the integrand*, and the variable x is *integration variable.*

The integration variable is a *dummy variable.* If you systematically replace it with another variable, the resulting integral will still be the same. For instance,

$$\int_0^1 x^2\, dx = \big[\tfrac{1}{3}x^3\big]_{x=0}^{1} = \tfrac{1}{3},$$

and if you replace x by φ you still get

$$\int_0^1 \varphi^2\, d\varphi = \big[\tfrac{1}{3}\varphi^3\big]_{\varphi=0}^{1} = \tfrac{1}{3}.$$

Another way to appreciate that the integration variable is a dummy variable is to look at the Fundamental Theorem again:

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The right hand side tells you that the value of the integral depends on a and b, and does not anything to do with the variable x.

Exercises

52.1 – Find an antiderivative $F(x)$ for each of the following functions $f(x)$. Finding antiderivatives involves a fair amount of guess work, but with experience it gets easier to guess antiderivatives.

(**a**) $f(x) = 2x + 1$	(**b**) $f(x) = 1 - 3x$
(**c**) $f(x) = x^2 - x + 11$	(**d**) $f(x) = x^4 - x^2$
(**e**) $f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$	(**f**) $f(x) = \frac{1}{x}$
(**g**) $f(x) = e^x$	(**h**) $f(x) = \frac{2}{x}$
(**i**) $f(x) = e^{2x}$	(**j**) $f(x) = \frac{1}{2 + x}$
(**k**) $f(x) = \frac{e^x - e^{-x}}{2}$	(**l**) $f(x) = \frac{1}{1 + x^2}$
(**m**) $f(x) = \frac{e^x + e^{-x}}{2}$	(**n**) $f(x) = \frac{1}{\sqrt{1 - x^2}}$
(**o**) $f(x) = \sin x$	(**p**) $f(x) = \frac{2}{1 - x}$
(**q**) $f(x) = \cos x$	(**r**) $f(x) = \cos 2x$
(**s**) $f(x) = \sin(x - \pi/3)$	(**t**) $f(x) = \sin x + \sin 2x$
(**u**) $f(x) = \sin(x) + \frac{1}{x}$	(**v**) $f(x) = 2x(1 + x^2)^5$

In each of the following exercises you should compute the area of the indicated region, and also of the smallest enclosing rectangle with horizontal and vertical sides.

Before computing anything draw the region.

52.2 – The region between the vertical lines $x = 0$ and $x = 1$, and between the x-axis and the graph of $y = x^3$.

52.3 – The region between the vertical lines $x = 0$ and $x = 1$, and between the x-axis and the graph of $y = x^n$ (here $n > 0$, draw for $n = \frac{1}{2}, 1, 2, 3, 4$).

52.4 – The region above the graph of $y = \sqrt{x}$, below the line $y = 2$, and between the vertical lines $x = 0$, $x = 4$.

52.5 – The region above the x-axis and below the graph of $f(x) = x^2 - x^3$.

52.6 – The region above the x-axis and below the graph of $f(x) = 4x^2 - x^4$.

52.7 – The region above the x-axis and below the graph of $f(x) = 1 - x^4$.

52.8 – The region above the x-axis, below the graph of $f(x) = \sin x$, and between $x = 0$ and $x = \pi$.

52.9 – The region above the x-axis, below the graph of $f(x) = 1/(1 + x^2)$ (a curve known as *Maria Agnesi's witch*), and between $x = 0$ and $x = 1$.

52.10 – The region between the graph of $y = 1/x$ and the x-axis, and between $x = a$ and $x = b$ (here $0 < a < b$ are constants, e.g. choose $a = 1$ and $b = \sqrt{2}$ if you have something against either letter a or b.)

52.11 – The region above the x-axis and below the graph of

$$f(x) = \frac{1}{1 + x} + \frac{x}{2} - 1.$$

52.12 – Compute

$$\int_0^1 \sqrt{1-x^2}dx$$

without find an antiderivative for $\sqrt{1-x^2}$ (you can find such an antiderivative, but it's not easy. This integral is the area of some region: which region is it, and what is that area?)

52.13 – Compute

$$\text{(a)} \int_0^{1/2} \sqrt{1-x^2}dx \qquad \text{(b)} \int_{-1}^1 |1-x|dx \qquad \text{(c)} \int_{-1}^1 |2-x|dx$$

without finding antiderivatives.

53. The indefinite integral

The fundamental theorem tells us that in order to compute the integral of some function f over an interval $[a, b]$ you should first find an antiderivative F of f. In practice, much of the effort required to find an integral goes into finding the antiderivative. In order to simplify the computation of the integral

$$\int_a^b f(x)dx = F(b) - F(a) \tag{46}$$

the following notation is commonly used for the antiderivative:

$$F(x) = \int f(x)dx. \tag{47}$$

For instance,

$$\int x^2\,dx = \frac{1}{3}x^3, \qquad \int \sin 5x\,dx = -\tfrac{1}{5}\cos 5x, \qquad \text{etc}\ldots$$

The integral which appears here does not have the integration bounds a and b. It is called an ***indefinite integral***, as opposed to the integral in (46) which is called a ***definite integral***. You use the indefinite integral if you expect the computation of the antiderivative to be a lengthy affair, and you do not want to write the integration bounds a and b all the time.

It is important to distinguish between the two kinds of integrals. Here is a list of differences:

Indefinite integral	Definite integral
$\int f(x)dx$ is a function of x.	$\int_a^b f(x)dx$ is a number.
By definition $\int f(x)dx$ is *any function of x whose derivative is $f(x)$*.	$\int_a^b f(x)dx$ was defined in terms of Riemann sums and can be interpreted as "area under the graph of $y = f(x)$", at least when $f(x) > 0$.
x is not a dummy variable, for example, $\int 2xdx = x^2+C$ and $\int 2tdt = t^2+C$ are functions of diffferent variables, so they are not equal.	x is a dummy variable, for example, $\int_0^1 2xdx = 1$, and $\int_0^1 2tdt = 1$, so $\int_0^1 2xdx = \int_0^1 2tdt$.

53.1. *You can always check the answer*

Suppose you want to find an antiderivative of a given function $f(x)$ and after a long and messy computation which you don't really trust you get an "answer", $F(x)$. You can then throw away the dubious computation and differentiate the $F(x)$ you had found. If $F'(x)$ turns out to be equal to $f(x)$, then your $F(x)$ is indeed an antiderivative and your computation isn't important anymore.

For example, suppose that we want to find $\int \ln x\, dx$. My cousin Louie says it might be $F(x) = x \ln x - x$. Let's see if he's right:

$$\frac{d}{dx}\big(x \ln x - x\big) = x \cdot \frac{1}{x} + 1 \cdot \ln x - 1 = \ln x.$$

Who knows how Louie thought of this[7], but it doesn't matter: he's right! We now know that $\int \ln x dx = x \ln x - x + C$.

53.2. *About "+C"*

Let $f(x)$ be a function defined on some interval $a \le x \le b$. If $F(x)$ is an antiderivative of $f(x)$ on this interval, then for any constant C the function $\tilde{F}(x) = F(x)+C$ will also be an antiderivative of $f(x)$. So one given function $f(x)$ has many different antiderivatives, obtained by adding different constants to one given antiderivative.

Theorem 53.1. *If $F_1(x)$ and $F_2(x)$ are antiderivatives of the same function $f(x)$ on some interval $a \le x \le b$, then there is a constant C such that $F_1(x) = F_2(x) + C$.*

Proof. Consider the difference $G(x) = F_1(x) - F_2(x)$. Then $G'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$, so that $G(x)$ must be constant. Hence $F_1(x) - F_2(x) = C$ for some constant. □

It follows that there is some ambiguity in the notation $\int f(x)\, dx$. Two functions $F_1(x)$ and $F_2(x)$ can both equal $\int f(x)\, dx$ without equaling each other. When this happens, they (F_1 and F_2) differ by a constant. This can sometimes lead to confusing situations, e.g. you can check that

$$\int 2 \sin x \cos x\, dx = \sin^2 x$$

$$\int 2 \sin x \cos x\, dx = -\cos^2 x$$

are both correct. (Just differentiate the two functions $\sin^2 x$ and $-\cos^2 x$!) These two answers look different until you realize that because of the trig identity $\sin^2 x + \cos^2 x = 1$ they really only differ by a constant: $\sin^2 x = -\cos^2 x + 1$.

To avoid this kind of confusion we will from now on never forget to include the "arbitrary constant $+C$" in our answer when we compute an antiderivative.

[7]He took math 222 and learned to integrate by parts.

53.3. *Standard Integrals*

Here is a list of the standard integrals everyone should know.

$$\int f(x)\,dx = F(x) + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \text{for all } n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \text{(Note the absolute values)}$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad \text{(don't memorize: use } a^x = e^{x\ln a}\text{)}$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C \qquad \text{(Note the absolute values)}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

The following integral is also useful, but not as important as the ones above:

$$\int \frac{dx}{\cos x} = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

All of these integrals should be familiar from the differentiation rules we have learned so far, except for for the integrals of $\tan x$ and of $\frac{1}{\cos x}$. You can check those by differentiation (using $\ln\frac{a}{b} = \ln a - \ln b$ simplifies things a bit).

54. **Properties of the Integral**

Just as we had a list of properties for the limits and derivatives of sums and products of functions, the integral has similar properties.

Suppose we have two functions $f(x)$ and $g(x)$ with antiderivatives $F(x)$ and $G(x)$, respectively. Then we know that

$$\frac{d}{dx}\bigl\{F(x) + G(x)\bigr\} = F'(x) + G'(x) = f(x) + g(x),$$

in words, $F + G$ is an antiderivative of $f + g$, which we can write as

$$\int \bigl\{f(x) + g(x)\bigr\}\,dx = \int f(x)\,dx + \int g(x)\,dx. \tag{48}$$

Similarly, $\frac{d}{dx}\bigl(cF(x)\bigr) = cF'(x) = cf(x)$ implies that

$$\int cf(x)\,dx = c\int f(x)\,dx \tag{49}$$

if c is a constant.

These properties imply analogous properties for the definite integral. For any pair of functions on an interval $[a, b]$ one has

$$\int_a^b \big[f(x) + g(x)\big]dx = \int_a^b f(x)dx + \int_a^b g(x)dx, \tag{50}$$

and for any function f and constant c one has

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx. \tag{51}$$

Definite integrals have one other property for which there is no analog in indefinite integrals: if you split the interval of integration into two parts, then the integral over the whole is the sum of the integrals over the parts. The following theorem says it more precisely.

Theorem 54.1. *Given $a < c < b$, and a function on the interval $[a, b]$ then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{52}$$

Proof. Let F be an antiderivative of f. Then

$$\int_a^c f(x)dx = F(c) - F(a) \text{ and } \int_c^b f(x)dx = F(b) - F(a),$$

so that

$$\begin{aligned}\int_a^b f(x)dx &= F(b) - F(a)\\ &= F(b) - F(c) + F(c) - F(a)\\ &= \int_a^c f(x)dx + \int_c^b f(x)dx.\end{aligned}$$

□

So far we have always assumed theat $a < b$ in all indefinitie integrals $\int_a^b \dots$. The fundamental theorem suggests that when $b < a$, we should define the integral as

$$\int_a^b f(x)dx = F(b) - F(a) = -\big(F(a) - F(b)\big) = -\int_b^a f(x)dx. \tag{53}$$

For instance,

$$\int_1^0 xdx = -\int_0^1 xdx = -\tfrac{1}{2}.$$

55. The definite integral as a function of its integration bounds

Consider the expression

$$I = \int_0^x t^2\, dt.$$

What does I depend on? To see this, you calculate the integral and you find

$$I = \big[\tfrac{1}{3}t^3\big]_0^x = \tfrac{1}{3}x^3 - \tfrac{1}{3}0^3 = \tfrac{1}{3}x^3.$$

So the integral depends on x. It does not depend on t, since t is a "dummy variable" (see §52.1 where we already discussed this point.)

In this way you can use integrals to define new functions. For instance, we could define

$$I(x) = \int_0^x t^2 dt,$$

which would be a roundabout way of defining the function $I(x) = x^3/3$. Again, since t is a dummy variable we can replace it by any other variable we like. Thus

$$I(x) = \int_0^x \alpha^2 \, d\alpha$$

defines the same function (namely, $I(x) = \frac{1}{3}x^3$).

The previous example does not define a new function ($I(x) = x^3/3$). An example of a *new* function defined by an integral is the "error-function" from statistics. It is given by

$$\operatorname{erf}(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt, \tag{54}$$

so $\operatorname{erf}(x)$ is the area of the shaded region in figure 32. The integral in (54) cannot be com-

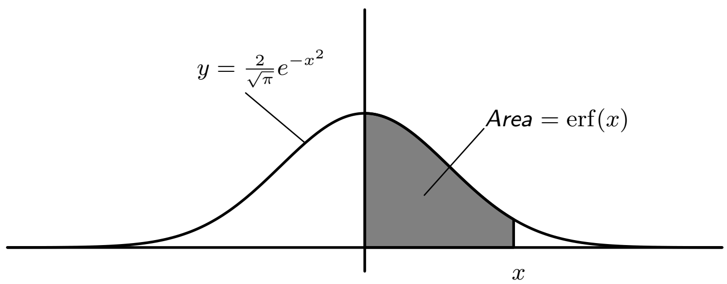


FIGURE 32. Definition of the Error function.

puted in terms of the standard functions (square and higher roots, sine, cosine, exponential and logarithms). Since the integral in (54) occurs very often in statistics (in relation with the so-called normal distribution) it has been given a name, namely, "$\operatorname{erf}(x)$".

How do you differentiate a function that is defined by an integral? The answer is simple, for if $f(x) = F'(x)$ then the fundamental theorem says that

$$\int_a^x f(t) \, dt = F(x) - F(a),$$

and therefore

$$\frac{d}{dx} \int_a^x f(t) \, dt = \frac{d}{dx} \big\{ F(x) - F(a) \big\} = F'(x) = f(x),$$

i.e.

$$\frac{d}{dx} \int_a^x f(t) \, dt = f(x).$$

A similar calculation gives you

$$\frac{d}{dx} \int_x^b f(t) \, dt = -f(x).$$

So what is the derivative of the error function? We have

$$\begin{aligned} \operatorname{erf}'(x) &= \frac{d}{dx} \Big\{ \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \Big\} \\ &= \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^x e^{-t^2} \, dt \\ &= \frac{2}{\sqrt{\pi}} e^{-x^2}. \end{aligned}$$

56. **Method of substitution**

The chain rule says that

$$\frac{dF(G(x))}{dx} = F'(G(x)) \cdot G'(x),$$

so that

$$\int F'(G(x)) \cdot G'(x)\, dx = F(G(x)) + C.$$

56.1. *Example*

Consider the function $f(x) = 2x\sin(x^2+3)$. It does not appear in the list of standard antiderivatives we know by heart. But we do notice[8] that $2x = \frac{d}{dx}(x^2+3)$. So let's call $G(x) = x^2+3$, and $F(u) = -\cos u$, then

$$F(G(x)) = -\cos(x^2+3)$$

and

$$\frac{dF(G(x))}{dx} = \underbrace{\sin(x^2+3)}_{F'(G(x))} \cdot \underbrace{2x}_{G'(x)} = f(x),$$

so that

$$\int 2x\sin(x^2+3)\, dx = -\cos(x^2+3) + C. \tag{55}$$

56.2. *Leibniz' notation for substitution*

The most transparent way of computing an integral by substitution is by following Leibniz and introduce new variables. Thus to do the integral

$$\int f(G(x))G'(x)\, dx$$

where $f(u) = F'(u)$, we introduce the substitution $u = G(x)$, and agree to write

$$du = dG(x) = G'(x)\, dx.$$

Then we get

$$\int f(G(x))G'(x)\, dx = \int f(u)\, du = F(u) + C.$$

At the end of the integration we must remember that u really stands for $G(x)$, so that

$$\int f(G(x))G'(x)\, dx = F(u) + C = F(G(x)) + C.$$

As an example, let's do the integral (55) using Leibniz' notation. We want to find

$$\int 2x\sin(x^2+3)\, dx$$

and decide to substitute $z = x^2+3$ (the substitution variable doesn't always have to be called u). Then we compute

$$dz = d\big(x^2+3\big) = 2x\, dx \text{ and } \sin(x^2+3) = \sin z,$$

so that

$$\int 2x\sin(x^2+3)\, dx = \int \sin z\, dz = -\cos z + C.$$

Finally we get rid of the substitution variable z, and we find

$$\int 2x\sin(x^2+3)\, dx = -\cos\big(x^2+3\big) + C.$$

---

[8] You *will* start noticing things like this after doing several examples.

When we do integrals in this calculus class, we always get rid of the substitution variable because it is a variable we invented, and which does not appear in the original problem. But if you are doing an integral which appears in some longer discussion of a real-life (or real-lab) situation, then it may be that the substitution variable actually has a meaning (e.g. "the effective stoichiometric modality of CQF self-inhibition") in which case you may want to skip the last step and leave the integral in terms of the (meaningful) substitution variable.

56.3. *Substitution for definite integrals*

For definite integrals the chain rule

$$\frac{d}{dx}\big(F(G(x))\big) = F'(G(x))G'(x) = f(G(x))G'(x)$$

implies

$$\int_a^b f(G(x))G'(x)\,dx = F(G(b)) - F(G(a)).$$

which you can also write as

$$\int_{x=a}^{b} f(G(x))G'(x)\,dx = \int_{u=G(a)}^{G(b)} f(u)\,du. \tag{56}$$

56.4. *Example of substitution in a definite integral*

Let's compute

$$\int_0^1 \frac{x}{1+x^2}\,dx,$$

using the substitution $u = G(x) = 1 + x^2$. Since $du = 2x\,dx$, the associated ***indefinite*** integral is

$$\int \underbrace{\frac{1}{1+x^2}}_{\frac{1}{u}} \underbrace{x\,dx}_{\frac{1}{2}du} = \tfrac{1}{2}\int \frac{1}{u}\,du.$$

To find the definite integral you must compute the new integration bounds $G(0)$ and $G(1)$ (see equation (56).) If x runs between $x = 0$ and $x = 1$, then $u = G(x) = 1 + x^2$ runs between $u = 1 + 0^2 = 1$ and $u = 1 + 1^2 = 2$, so the definite integral we must compute is

$$\int_0^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du, \tag{57}$$

which is in our list of memorable integrals. So we find

$$\int_0^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du = \tfrac{1}{2}\big[\ln u\big]_1^2 = \tfrac{1}{2}\ln 2.$$

Sometimes the integrals in (57) are written as

$$\int_{x=0}^1 \frac{x}{1+x^2}\,dx = \tfrac{1}{2}\int_{u=1}^2 \frac{1}{u}\,du,$$

to emphasize (and remind yourself) to which variable the bounds in the integral refer.

Exercises

56.1 – Compute these derivatives

(a) $\dfrac{d}{dx}\displaystyle\int_0^x \left(1+t^2\right)^4\, dt$

(b) $\dfrac{d}{dx}\displaystyle\int_x^1 \ln z\, dz$

(c) $\dfrac{d}{ds}\displaystyle\int_s^0 \frac{dx}{1+x^2}$

(d) $\dfrac{d}{dx}\displaystyle\int_x^{2x} s^2 ds$

(e) $\dfrac{d}{d\theta}\displaystyle\int_0^{\sin\theta} \frac{dx}{1-x^2}$

(f) $\dfrac{d}{dt}\displaystyle\int_0^{t^2} e^{2x} dx$

56.2 – Compute the second derivative of the error function. How many inflection points does the graph of the error function have?

Compute (some of) the following integrals

56.3 – $\displaystyle\int \left\{6x^5 - 2x^{-4} - 7x + 3/x - 5 + 4e^x + 7^x\right\} dx$

56.4 – $\displaystyle\int (x/a + a/x + x^a + a^x + ax)\, dx$

56.5 – $\displaystyle\int \left\{\sqrt{x} - \sqrt[3]{x^4} + \frac{7}{\sqrt[3]{x^2}} - 6e^x + 1\right\} dx$

56.6 – $\displaystyle\int \left\{2^x + \left(\tfrac{1}{2}\right)^x\right\} dx$

56.7 – $\displaystyle\int_{-2}^4 (3x-5)\, dx$

56.8 – $\displaystyle\int_1^4 x^{-2}\, dx$ (hm. . .)

56.9 – $\displaystyle\int_1^4 t^{-2}\, dt$ **(!)**

56.10 – $\displaystyle\int_1^4 x^{-2}\, dt$ **(!!!)**

56.11 – $\displaystyle\int_0^1 (1-2x-3x^2)\, dx$

56.12 – $\displaystyle\int_1^2 (5x^2-4x+3)\, dx$

56.13 – $\displaystyle\int_{-3}^0 (5y^4-6y^2+14)\, dy$

56.14 – $\displaystyle\int_0^1 (y^9-2y^5+3y)\, dy$

56.15 – $\displaystyle\int_0^4 \sqrt{x}\, dx$

56.16 – $\displaystyle\int_0^1 x^{3/7}\, dx$

56.17 – $\displaystyle\int_1^3 \left(\frac{1}{t^2}-\frac{1}{t^4}\right) dt$

56.18 – $\displaystyle\int_1^2 \frac{t^6-t^2}{t^4}\, dt$

56.19 – $\displaystyle\int_1^2 \frac{x^2+1}{\sqrt{x}}\, dx$

56.20 – $\displaystyle\int_0^2 (x^3-1)^2\, dx$

56.21 – $\displaystyle\int_0^1 u(\sqrt{u}+\sqrt[3]{u})\, du$

56.22 – $\displaystyle\int_1^2 (x+1/x)^2\, dx$

56.23 – $\displaystyle\int_3^3 \sqrt{x^5+2}\, dx$

56.24 – $\displaystyle\int_1^{-1} (x-1)(3x+2)\, dx$

56.25 – $\displaystyle\int_1^4 (\sqrt{t}-2/\sqrt{t})\, dt$

56.26 – $\displaystyle\int_1^8 \left(\sqrt[3]{r}+\frac{1}{\sqrt[3]{r}}\right) dr$

56.27 – $\displaystyle\int_{-1}^0 (x+1)^3\, dx$

56.28 – $\displaystyle\int_{-5}^{-2} \frac{x^4-1}{x^2+1}\, dx$

56.29 – $\displaystyle\int_1^e \frac{x^2+x+1}{x}\, dx$

56.30 – $\displaystyle\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

56.31 – $\displaystyle\int_0^1 \left(\sqrt[4]{x^5}+\sqrt[5]{x^4}\right) dx$

56.32 – $\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}\,dx$

56.33 – $\int_{\pi/4}^{\pi/3} \sin t\,dt$

56.34 – $\int_0^{\pi/2} (\cos\theta + 2\sin\theta)\,d\theta$

56.35 – $\int_0^{\pi/2} (\cos\theta + \sin 2\theta)\,d\theta$

56.36 – $\int_{2\pi/3}^{\pi} \frac{\tan x}{\cos x}\,dx$

56.37 – $\int_{\pi/3}^{\pi/2} \frac{\cot x}{\sin x}\,dx$

56.38 – $\int_1^{\sqrt{3}} \frac{6}{1+x^2}\,dx$

56.39 – $\int_0^{0.5} \frac{dx}{\sqrt{1-x^2}}$

56.40 – $\int_4^8 (1/x)\,dx$

56.41 – $\int_{\ln 3}^{\ln 6} 8e^x\,dx$

56.42 – $\int_8^9 2^t\,dt$

56.43 – $\int_{-e^2}^{-e} \frac{3}{x}\,dx$

56.44 – $\int_{-2}^{3} |x^2-1|\,dx$

56.45 – $\int_{-1}^{2} |x-x^2|\,dx$

56.46 – $\int_{-1}^{2} (x-2|x|)\,dx$

56.47 – $\int_0^2 (x^2-|x-1|)\,dx$

56.48 – $\int_0^2 f(x)\,dx$ where

$$f(x) = \begin{cases} x^4 & \text{if } 0 \le x < 1, \\ x^5, & \text{if } 1 \le x \le 2. \end{cases}$$

56.49 – $\int_{-\pi}^{\pi} f(x)\,dx$ where

$$f(x) = \begin{cases} x, & \text{if } -\pi \le x \le 0, \\ \sin x, & \text{if } 0 < x \le \pi. \end{cases}$$

56.50 – Compute

$$I = \int_0^2 2x\big(1+x^2\big)^3\,dx$$

in two different ways:
(**a**) Expand $(1+x^2)^3$, multiply with $2x$, and integrate each term.
(**b**) Use the substitution $u = 1+x^2$.

56.51 – Compute

$$I_n = \int 2x\big(1+x^2\big)^n\,dx.$$

56.52 – If $f'(x) = x - 1/x^2$ and $f(1) = 1/2$ find $f(x)$.

56.53 – Sketch the graph of the curve $y = \sqrt{x+1}$ and determine the area of the region enclosed by the curve, the x-axis and the lines $x = 0$, $x = 4$.

56.54 – Find the area under the curve $y = \sqrt{6x+4}$ and above the x-axis between $x = 0$ and $x = 2$. Draw a sketch of the curve.

56.55 – Graph the curve $y = 2\sqrt{1-x^2}$, $x \in [0,1]$, and find the area enclosed between the curve and the x-axis. (Don't evaluate the integral, but compare with the area under the graph of $y = \sqrt{1-x^2}$.)

56.56 – Determine the area under the curve $y = \sqrt{a^2-x^2}$ and between the lines $x = 0$ and $x = a$.

56.57 – Graph the curve $y = 2\sqrt{9-x^2}$ and determine the area enclosed between the curve and the x-axis.

56.58 – Graph the area between the curve $y^2 = 4x$ and the line $x = 3$. Find the area of this region.

56.59 – Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0$ and $y = 3$.

56.60 – Find the area enclosed between the curve $y = \sin 2x$, $0 \le x \le \pi/4$ and the axes.

56.61 – Find the area enclosed between the curve $y = \cos 2x$, $0 \le x \le \pi/4$ and the axes.

56.62 – Graph $y^2 + 1 = x$, and find the area enclosed by the curve and the line $x = 2$.

56.63 – Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

56.64 – Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.

56.65 – Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

56.66 – Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

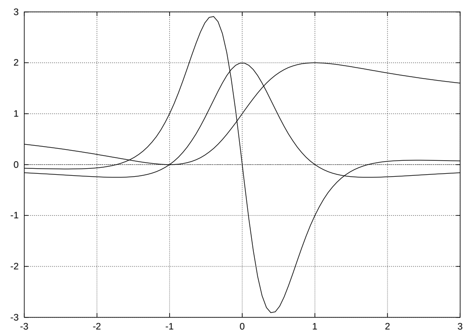
56.67 – Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

56.68 – Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

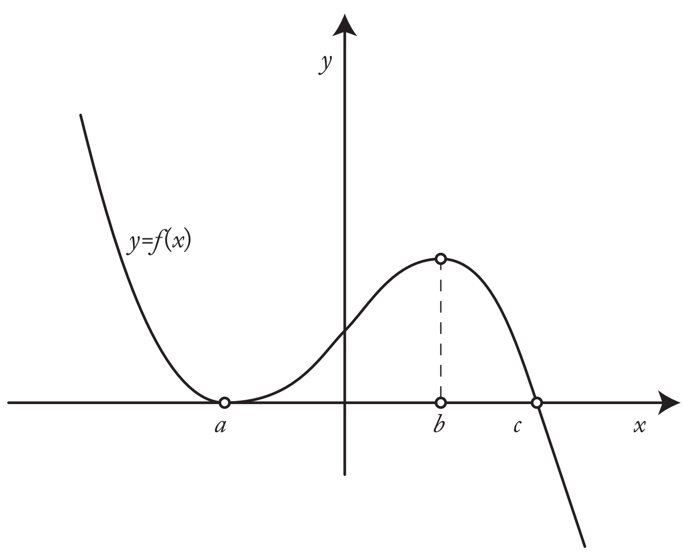
56.69 – Here are the graphs of a function $f(x)$, its derivative $f'(x)$ and an antiderivative $F(x)$ of $f(x)$.

Unfortunately the graphs are not labeled. Identify which graph is which.

Explain your answer.



56.70 – Below is the graph of a function $y = f(x)$



Which among the following statements true?

(**b**) $F(a) = F(c)$? *True/False* Reason:
(**c**) $F(b) = 0$? *True/False* Reason:
(**d**) $F(b) > F(c)$? *True/False* Reason:
(**e**) The graph of $y = F(x)$ has ***two*** inflection points? *True/False* Reason:

Use a substitution to evaluate the following integrals.

56.71 – $\int_1^2 \frac{u\,du}{1+u^2}$

56.72 – $\int_0^5 \frac{x\,dx}{\sqrt{x+1}}$

56.73 – $\int_1^2 \frac{x^2\,dx}{\sqrt{2x+1}}$

56.74 – $\int_0^5 \frac{s\,ds}{\sqrt[3]{s+2}}$

56.75 – $\int_1^2 \frac{x\,dx}{1+x^2}$

56.76 – $\int_0^{\pi} \cos(\theta + \frac{\pi}{3})d\theta$

56.77 – $\int \sin\frac{\pi+x}{5}dx$

56.78 – $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\,dx$

56.79 – $\int_{\pi/4}^{\pi/3} \sin^2\theta\cos\theta\,d\theta$

56.80 – $\int_2^3 \frac{1}{r\ln r}, dr$

56.81 – $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

56.82 – $\int \frac{\sin 2x}{1+\sin x}\,dx$

56.83 – $\int_0^1 z\sqrt{1-z^2}\,dz$

56.84 – $\int_1^2 \frac{\ln 2x}{x}\,dx$

56.85 – $\int_{\xi=0}^{\sqrt{2}} \xi(1+2\xi^2)^{10}\,d\xi$

56.86 – $\int_2^3 \sin\rho\big(\cos 2\rho\big)^4\,d\rho$

56.87 – $\int \alpha e^{-\alpha^2}\,d\alpha$

56.88 – $\int \frac{e^{\frac{1}{t}}}{t^2}\,dt$