

EXERCISES

49.1 - Sketch the graphs of the following functions.

- (a)  $y = e^x$   
 (b)  $y = e^{-x}$   
 (c)  $y = e^x + e^{-2x}$ .  $dy/dx = e^x - 2e^{-2x}$ . Local min at  $x = \frac{1}{3} \ln 2$ . The function is convex.  
 (d)  $y = e^{3x} - 4e^x$ .  $dy/dx = 3e^{3x} - 4e^x$ . local min at  $x = \frac{1}{2} \ln 43$ .  $d^2y/dx^2 = 9e^{3x} - 4e^x$ . Inflection point at  $x = \frac{1}{2} \ln \frac{4}{9}$ .  
 subproby  $= \frac{e^x}{1+e^x}$ .  $dy/dx = e^x/(1+e^x)^2 > 0$  function is increasing.  $d^2y/dx^2 = (e^x - e^{2x})/(1+e^x)^3$ . Inflection oint at  $x = 0$ . Limits:  $\lim_{x \rightarrow \infty} y = 1$ ;  $\lim_{x \rightarrow -\infty} y = 0$ .  
 (e)  $y = \frac{2e^x}{1+e^{2x}}$   
 (f)  $y = xe^{-x}$ .  $dy/dx = (1-x)e^{-x}$ . Absolute max at  $x = 1$ .  $d^2y/dx^2 = (x-2)e^{-x}$ , inflection point at  $x = 2$ . Limits:  $\lim_{x \rightarrow \infty} y = 0$ ;  $\lim_{x \rightarrow -\infty} y = -\infty$  (DNE).  
 (g)  $y = \sqrt{x}e^{-x/4}$ .  $dy/dx = (\frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{4})e^{-x/4} = \frac{2-x}{4\sqrt{x}}e^{-x/4}$ . Global max at  $x = 2$ .  $d^2y/dx^2 = (-4 - 4x + x^2)e^{-x/4}/x^{3/2}$ ; inflection point at  $x = 2 + 2\sqrt{2}$ .  
 (h)  $y = x^2e^{x+2}$   
 (i)  $y = e^{x/2} - x$ . Function is convex. Has global min at  $x = 2 \ln 2$

49.2 - Sketch the graphs of the following functions. (a)  $y = \ln \sqrt{x} = \frac{1}{2} \ln x$

- (b)  $y = \ln \frac{1}{x} = -\ln x$   
 (c)  $y = x \ln x$ .  $dy/dx = \ln x + 1$ . Global min when  $\ln x = -1$ , i.e. when  $x = 1/e = e^{-1}$ .  $d^2y/dx^2 = 1/x > 0$ . Function is convex.  
 (d)  $y = \frac{-1}{\ln x}$  ( $0 < x < \infty, x \neq 1$ )  
 (e)  $y = (\ln x)^2$  ( $x > 0$ ).  $dy/dx = 2(\ln x)/x$ ,  $d^2y/dx^2 = 2(1 - \ln x)/x^2$ . Global minimum at  $x = 1$ , inflection point at  $x = e$ .  
 (f)  $y = \frac{\ln x}{x}$  ( $x > 0$ )  
 (g)  $y = \ln \sqrt{\frac{1+x}{1-x}} = \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x)$  ( $|x| < 1$ ) Second form is easier to differentiate.  
 (h)  $y = \ln(1+x^2)$ .  $dy/dx = 2x/(1+x^2)$ ,  $d^2y/dx^2 = 2(1-x^2)/(1+x^2)^2$ . Global min at  $x = 0$ , inflection points at  $x = \pm 1$ .  
 (i)  $y = \ln(x^2 - 3x + 2)$  ( $x > 2$ )  
 (j)  $y = \ln \cos x$  ( $|x| < \frac{\pi}{2}$ )  $dy/dx = \tan x$ ,  $d^2y/dx^2 = 1/\cos^2 x > 0$ . Function is convex, has global min at  $x = 0$ .

49.3 - Find the inflection points on the graph of  $f(x) = (1+x) \ln x$  ( $x > 0$ ).

Derivatives:

$$f'(x) = \ln x + \frac{1}{x} + 1, \quad f''(x) = \frac{1}{x} - \frac{1}{x^2}$$

You can't solve  $f'(x) = 0$ , but that's not asked. We want to find where  $f''(x) = 0$ . Inflection point at  $x = 1$ .

49.4 - Find the tenth derivative of  $f(x) = xe^x$ .

The derivatives are  $f'(x) = (x+1)e^x$ ,  $f^{(2)}(x) = (x+2)e^x$ , ...,  $f^{(10)}(x) = (x+10)e^x$ .

49.5 - For which real number  $x$  is  $2^x - 3^x$  the largest?

$f(x) = 2^x - 3^x = e^{x \ln 2} - e^{x \ln 3}$ .  $f'(x) = (\ln 2)2^x - (\ln 3)3^x$ . Global max when

$$(\ln 2)2^x = (\ln 3)3^x.$$

2

Take natural logs:

$$\ln \ln 2 + x \ln 2 = \ln \ln 3 + x \ln 3 \implies x = \frac{\ln \ln 3 - \ln \ln 2}{\ln 3 - \ln 2}.$$