Math 221 – Notes on Rolle's Theorem, The Mean Value Theorem, l'Hôpital's rule, and the Taylor-Maclaurin formula

1. Two theorems

Rolle's Theorem. *If a function $y = f(x)$ is differentiable for $a \le x \le b$ and if $f(a) = f(b) = 0$, then there is a number $a < c < b$ such that $f'(c) = 0$.*

Exercise. Suppose $y = f(x)$ is a twice differentiable function. Suppose also that there are three different numbers $a < b < c$ such that $f(a) = f(b) = f(c) = 0$. Then show that there is a number x between a and b such that $f''(x) = 0$. (Done in class.)

The Mean Value Theorem. *If a function $y = f(x)$ is differentiable for $a \le x \le b$ then there is a number $a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{1}$$

holds.

Exercise. Suppose $y = f(x)$ is a differentiable function on some interval $a \le x \le b$. Suppose also that $f'(x) > 0$ for all x between a and b. Then $f(a) < f(b)$.

(Solution: By the MVThm there is a c between a and b with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We know that $f'(c) > 0$, and also that $b - a > 0$. Therefore

$$f(b) - f(a) = f'(c)(b - a) > 0.)$$

2. l'Hôpital's Rule

Let $y = f(x)$ be a differentiable function. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{2}$$

More precisely, if the limit on the right exists, then so does the limit on the left, and both are then equal.

Here is a proof of the rule:

Let b be some number with $b \ne a$. Consider the function

$$h(x) = f(x)g(b) - f(b)g(x).$$

This function satisfies

$$h(a) = f(a)g(b) - f(b)g(a) = 0,$$

since both $f(a) = 0$ and $g(a) = 0$. It also satisfies

$$h(b) = f(b)g(b) - f(b)g(b) = 0.$$

We can therefore apply Rolle's theorem to $h(x)$, and conclude that there is a number c between a and b with $h'(c) = 0$. Now compute $h'(x)$:

$$h'(x) = f'(x)g(b) - f(b)g'(x).$$

Thus $h'(c) = 0$ implies

$$f'(c)g(b) - f(b)g'(c) = 0, \text{i.e. } \frac{f'(c)}{g'(c)} = \frac{f(b)}{g(b)}.$$

So far we have shown the following: for any $b \neq a$ you can find a c between a and b with $\frac{f'(c)}{g'(c)} = \frac{f(b)}{g(b)}$. As b tends towards a the corresponding number c must also converge towards a, since it is stuck between a and b. Hence

$$\lim_{b \to a} \frac{f(b)}{g(b)} = \lim_{c \to a} \frac{f'(c)}{g'(c)}.$$

So if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then so does $\lim_{x \to a} \frac{f(x)}{g(x)}$, and both limits must be equal.

Exercises & examples. See the book, §3-9. The book only gives examples where l'Hôpital's rule works. Here are two limits where the rule doesn't give you an answer.

$$\lim_{x \to 0} \frac{x + x^2 \sin(1/x)}{\sin x} \tag{3}$$

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{4}$$

The limit in (3) is equal to 1, but when you apply l'Hôpital you get

$$\lim_{x \to 0} \frac{1 + 2x \sin(1/x) - \cos(1/x)}{\cos x}$$

which does not exist (the denominator goes to 1, but the numerator oscillates up and down without converging to anything).

The limit in (4) is also 1 which you can see after dividing top and bottom by e^x, but if you apply l'Hôpital twice you get the same limit back – try it! (If you haven't seen exponentials in calculus yet, then wait until we've done thefunction $y = e^x$ later this semester.)

So, when you use l'Hôpital's rule you must always check that the limit is of the type $\frac{0}{0}$; if application of l'Hôpital's rule gives you a limit that "does not exist," then you can't use the rule, and you have to compute the limit some other way.

3. The Taylor-Maclaurin Formula

Question: Suppose $y = f(x)$ is a given function. Can you find a quadratic function $y = Q(x) = ax^2 + bx + c$ such that

$$f(0) = Q(0), \quad f'(0) = Q'(0), \quad f''(0) = Q''(0)$$

holds?

Answer: Yes, you compute $Q(0) = c$, $Q'(0) = b$ and $Q''(0) = 2a$, which gives you

$$Q(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2.$$

Instead of asking this question for quadratic polynomials, you could ask for a polynomial $P(x)$ of degree n whose first n derivatives at $x = 0$ are the same as those of the function $y = f(x)$. There is one such polynomial, and it is given by

$$P(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n. \tag{5}$$

This polynomial is called the n^{th} order Taylor-Maclaurin polynomial of the function $y = f(x)$.

You could ask for something even more general: Given a number a and a function $y = f(x)$, is there a polynomial $P(x)$ of degree n whose first n derivatives at $x = a$ coincide with those of $y = f(x)$? The answer is again yes, and P is given by

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{6}$$

4. The remainder term in Taylor's formula

The Taylor-Maclaurin polynomial $P(x)$ of a function $y = f(x)$ is almost never exactly equal to $f(x)$, but very often it is a good approximation to $f(x)$. There is a formula for the difference, and here it is:

If $P(x)$ is the Taylor Maclaurin polynomial for $y = f(x)$ at $x = 0$, (so $P(x)$ is given by (5)) then one has

$$f(x) = P(x) + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \tag{7}$$

where c is some number between 0 and x.

The proof of the remainder theorem uses Rolle's theorem (several times) and can be found in our text book in §3-10.

5. An example

Let $y = f(x) = \sin x$. one computes:

$$\begin{aligned}
f(x) &= \sin x & &\Rightarrow f(0) = 0 \\
f(x) &= \cos x & &\Rightarrow f'(0) = 1 \\
f(x) &= -\sin x & &\Rightarrow f''(0) = 0 \\
f(x) &= -\cos x & &\Rightarrow f^{(3)}(0) = -1 \\
f(x) &= \sin x & &\Rightarrow f^{(4)}(0) = 0 \\
&\vdots & &\vdots
\end{aligned}$$

So we see that the Taylor Maclaurin polynomial of $y = \sin x$ at $x = 0$ of order 4 is given by

$$P_4(x) = x - \frac{1}{6}x^3.$$

Note that in this example $f(0) = f''(0) = f^{(4)}(0) = 0$, so that there are no terms of order 0, 2 or 4 in the Taylor Maclaurin polynomial (their coefficients are zero).

Since $f^{(5)}(x) = \cos x$, the remainder term is given by

$$\sin x - P_4(x) = \frac{\cos c}{120}x^5.$$

All we know about c is that it lies between 0 and x, so we can't really compute $\cos c$, but we do know that $|\cos c| \le 1$, no matter what c is. Therefore the remainder satisfies

$$|\sin x - P_4(x)| \le \frac{|x^5|}{120}.$$

So for values of x with $|x| \le 0.1$, the polynomial $P_4(x)$ differs from $\sin x$ by at most

$$\frac{(0.1)^5}{120} < 0.000\,000\,1 \tag{8}$$

For $|x| < \frac{1}{2}$ the polynomial $P_4(x)$ approximates $\sin x$ with an error of at most

$$\frac{(1/2)^5}{120} \approx 0.000\,25$$

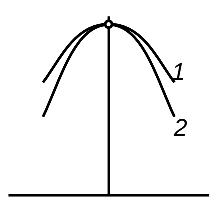
If you want to compute $\sin(0.1)$, then this gives you

$$\begin{aligned}\sin(0.1) &= P_4(0.1) + \text{Error} \\ &= 0.1 - \frac{(0.1)^3}{6} + \text{Error} \\ &= 0.1 - 0.000\,16666\ldots + \text{Error} \\ &= 0.099\,833\,333\ldots + \text{Error}\end{aligned}$$

in which we already have computed in (8) that the error is at most $0.000\,000\,1$. The actual value is $\sin(0.1) = 0.099\,833\,416\ldots$

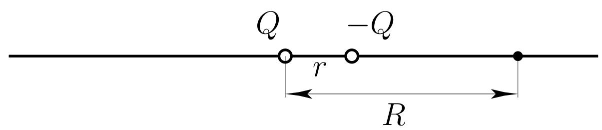
6. Problems on the Taylor Maclaurin formula

(1) Find the Taylor Maclaurin polynomials (at $a = 0$) of degrees 2, 3, 4, 5 and 6 of $f(x) = 1 + 2x - 7x^2 + \frac{1}{4}x^4 + x^5$.
(2) Compute the 12th degree Taylor Maclaurin polynomial (at $a = 0$) for $f(x) = \cos x$.
(3) Compute the 3rd degree Taylor Maclaurin polynomial (at $a = 0$) for $f(x) = \tan x$.
(4) Compute the 5th degree Taylor Maclaurin polynomial (at $a = 0$) for $f(x) = \frac{1}{(1+x)^2}$.
(5) Compute $\sqrt{10}$ the old fashioned way:
  (a) Compute the third degree Taylor Maclaurin polynomial of $f(x) = \sqrt{9 + x}$
  (b) Use the third degree Taylor Maclaurin polynomial to approximate $\sqrt{10}$ by setting $x = 1$.
  (c) Estimate the error in the approximation you got in part (b).
(6) Approximate $\sqrt[3]{9}$ in the same way as in the previous problem by finding the 4th degree Taylor Maclaurin polynomial of $f(x) = (8 + x)^{1/3}$, and estimating the error term.
(7) Here are graphs of $f(x) = \cos x$ and of $g(x) = 1 - x^2$. Which is which?



(8) (An example from physics.) A charged particle with charge Q induces an electric field E whose strength depends on the distance to the charge via the formula

$$E = \frac{Q}{R^2}.$$



The configuration in which you have two opposite charges of equal strengths ($+Q$ and $-Q$) close to each other is called a *dipole*.

If the positive charge is at the origin and the negative charge is at the point $x = r$, then the electric field at $x = R$ is given by

$$E = \frac{Q}{R^2} - \frac{Q}{(R-r)^2}.$$

This formula is often simplified by assuming r is small, and replacing $f(r) = \frac{Q}{(R-r)^2}$ by its first or sometimes second degree Taylor Maclaurin polynomial in r. *What do you get if you do this?*