The Method of Undetermined Coefficients

Our text contains, in section 18-10, one technique for finding a particular solution to a nonhomogeneous linear differential equation with constant coefficients, called Variation of Parameters. This is a powerful tool and you should definitely learn to use Variation of Parameters if you intend to go on in subjects where such equations come up frequently.

There is another technique which works to solve some such problems, called Undetermined Coefficients, which when it does work is generally much easier to use. In Math 222 I will only expect you to solve problems where this technique can be used: You are welcome to use the more powerful Variation of Parameters technique on these problems, but I am guaranteeing that any nonhomogeneous linear differential equation with constant coefficients that you need to solve for this class will be one that can be done using Undetermined Coefficients. The following introduction to the technique is adapted from a later edition of our text, the 8^{th} edition instead of the 5^{th}.

Before proceeding, remember what the goal is for either Undetermined Coefficients or Variation of Parameters, in our somewhat restricted circumstances: We learned how to find ***all*** solutions of an equation like

$$\alpha y'' + \beta y' + \gamma y = 0, \tag{hom}$$

which have constant coefficients α, β, γ. The text typically uses y_h for the general solution to this *homogeneous equation.* If what we really wanted to solve was an equation like this but with $f(x) \neq 0$, we learned that if we can find any one solution to the equation, we can combine that with the general solution to the homogeneous equation (the one we get by replacing $f(x)$ by 0) to get all solutions to the non-homogeneous equation. The book writes y_p for any *particular solution* to

$$\alpha y'' + \beta y' + \gamma y = f(x), \tag{inhom}$$

so this says we can write $y = y_h + y_p$ to represent all solutions. The problem we still have to solve is finding that particular solution y_p. Since ***any*** solution, any function that solves $\alpha y'' + \beta y' + \gamma = f(x)$, is OK, any way of finding y_p is fine, including a lucky guess. The first example done in the book, on page 875, is essentially done that way: When the text says "observe that $y =$ constant would do, provided ...", that amounts to a lucky guess. But since luck is not always with you, we need a more systematic way to find y_p, and that is what both *Variation of Parameters* and *Undetermined Coefficients* attempt to provide.

Suppose we start with an equation of the form (inhom). Our process for finding the general solution (y_h) to the corresponding homogeneous equation requires finding and solving the associated characteristic equation

$$\alpha r^2 + \beta r + \gamma = 0,$$

and using the resulting values of r in exponential and trigonometric functions. We will use the roots of that characteristic equation also in finding our particular solution y_p.

We will assume that $f(x)$ being simple means it is some combination of terms like e^{nx}, $\cos(kx)$, $\sin(kx)$, and polynomials $ax^2 + bx + c$. (Note that if both cosine and sine terms are present, if they have the same argument kx they can be treated as one. But if they have different arguments they must be treated separately, each resulting in a combination of sin and cosine terms in y_p.) Based on those terms we will put together a candidate y_p that has some constants in it we need to solve for: Those are the undetermined coefficients this method is named for.

An example. In what follows, refer to the table at the end of this handout. It will tell us what terms to combine for our trial solution to be used in finding y_p. How we use the table is best shown by examples: Here is Example 2 from page 876 in our text, but done with Undetermined Coefficients rather than Variation of Parameters. The task is to find all solutions of $y'' + 2y' - 3y = 6$. The characteristic equation is $r^2 + 2r - 3 = 0$, with roots $r = -3$ and $r = 1$. The right side of the equation in this case is just $f(x) = 6$, a polynomial of degree zero. The last row of the table tells us what to do when $f(x)$ contains a polynomial of degree ≤ 2. The second column says we need to check the roots of the characteristic equation: The roots of the characteristic equation were -3 and 1, so 0 is not a characteristic root. The third column says in this case we try for our particular solution y_p "a polynomial $Dx^2 + Ex + F$ of the same degree as $ax^2 + bx + c$." Since our polynomial was just the constant 6, a polynomial of degree 0, we use a polynomial of that same degree: I.e., we try for y_p a generic polynomial of degree zero, a constant. Let $y_p = F$ where F represents a constant yet to be determined. (I.e., F is the constant term of a polynomial Dx^2+Ex+F where, since we needed only degree 0, D and E are zero.) Then y_p' and y_p'' are both 0. Plugging this into the equation, for y_p to be a solution, we must have $0 + 0 - 3F = 6$. Hence $F = -2$ will make $y_p = F$ be a solution to the equation. (This also showed that $y_p = -2$ is the only solution that is just a constant, but all we needed was to find some solution. Note that this is exactly the particular solution arrived at by Variation of Parameters in the book: They would not have to be the same, since any other function which was obtained from this one by adding on a solution to the homogeneous equation would be another usable particular solution.) The complete solution is obtained by adding on the general solution $C_1e^{-3x} + C_2e^x$ to the homogeneous equation, getting $C_1e^{-3x} + C_2e^x - 2$ as the general solution to the nonhomogeneous equation.

Another example: Solve $y'' - y' = 5e^x - \sin(2x)$. The characteristic equation is $r^2 - r = 0$, with roots $r = 0$ and $r = 1$. The terms in $f(x)$ are $5e^x$ and $-\sin(2x)$. Start with $5e^x$, which is a constant times e^{nx} where $n = 1$. Since 1 is a single characteristic root, we put the term $Cx\,e^x$ into y_p. Now $-\sin(2x)$ is a multiple of $\sin(kx)$ where $k = 2$, and $2i$ is not a characteristic root, so we include $A\cos(2x) + B\sin(2x)$ in our candidate solution. Thus we arrive at $y_p = A\cos(2x)+B\sin(2x)+Cx\,e^x$. Then $y_p' = -2A\sin(2x)+2B\cos(2x)+Ce^x+Cx\,e^x$, and $y_p'' = -4A\cos(2x)-4B\sin(2x)+2Ce^x+Cx\,e^x$. Putting these into the equation we get $-4A\cos(2x)-4B\sin(2x)+2Ce^x+Cx\,e^x+2A\sin(2x)-2B\cos(2x)-Ce^x-Cx\,e^x = 5e^x - \sin(2x)$. We collect together the terms from both sides with $\cos(2x)$ and get $-4A - 2B = 0$. From the $\sin(2x)$ terms we get $-4B + 2A = -1$. The xe^x terms cancel out (a consequence of the fact that 1 was a root of the characteristic equation...) and the e^x terms give $C = 5$. Solving we get $A = -1/10$ and $B = 1/5$. Hence our particular solution is $y_p = -\frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x + 5xe^x$. Combining this with the general solution the homogeneous equation we get is $-\frac{1}{10}\cos 2x+\frac{1}{5}\sin 2x+5xe^x + C_1 + C_2e^x$.

Note that this method does have limitations. As given here, for example, it would not apply to Problem 2 on page 877 in the text, $y'' + y = \tan x$, since the limited table I included above does not suggest what to use for y_p when $f(x)$ includes the tangent function. It will, however, work for all other problems on that page and for the "story problems" on page 882.

For a term in $f(x)$ which is a multiple of	*If*	Then use a term like
$\sin(kx)$ or $\cos(kx)$	ki is not a characteristic root	$A\cos(kx) + B\sin(kx)$
	ki is a characteristic root	$Ax\cos(kx) + Bx\sin(kx)$
e^{nx}	n is not a characteristic root	Ce^{nx}
	n is a single characteristic root	$Cx\,e^{nx}$
	n is a double characteristic root	$Cx^2\,e^{nx}$
A polynomial $ax^2 + bx + c$ of degree at most 2	0 is not a characteristic root	a polynomial Dx^2+Ex+F of the same degree as ax^2+bx+c
	0 is a single characteristic root	a polynomial Dx^3+Ex^2+Fx of degree one more
	0 is a double characteristic root	a polynomial $Dx^4+Ex^3+Fx^2$ of degree two more