Sample vector and parametric curve questions for the final exam
Math 222, December 2006.

1

(i) Find the equation for the plane \( \mathcal{P} \) through the origin \( O \) and the points \( A(0,1,0) \) and \( B(1,1,12) \).

**Solution.** To get a normal vector to \( \mathcal{P} \) compute

\[
\vec{m} = \overrightarrow{OA} \times \overrightarrow{OB} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 12 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ -1 \end{pmatrix}.
\]

To find the equation for \( \mathcal{P} \) we pick a point on \( \mathcal{P} \). The origin \( O \) is the easiest point. Its position vector is the zero vector \( \vec{0} \), and so the equation for \( \mathcal{P} \) is

\[
\vec{m} \cdot \vec{x} = \vec{m} \cdot \vec{0},
\]

i.e. \( 12x_1 - x_3 = 12 \cdot 0 + 0 \cdot 0 + (-1) \cdot 0 = 0 \).

You could of course have chosen the points \( A \) or \( B \) instead of the origin. But both choices must give you the same equation \( 12x_1 - x_3 = 0 \) (check it!).

(ii) Find the vector representation of the line \( \ell \) through the points \( P(-1,2,2) \) and \( Q(-2,0,0) \). [“vector representation” is the same as “parametric equations”].

**Solution.** There are a few ways of writing this equation:

\[
\vec{x} = \vec{p} + t(\vec{q} - \vec{p}),
\]

or \( \vec{x} = \vec{p} + t\overrightarrow{PQ} \).

These are the same since \( \overrightarrow{PQ} = \vec{q} - \vec{p} \). The parameter doesn’t have to be called \( t \); it can be any letter you haven’t used yet. In this problem \( t \) is unused so far, so it will do.

You get

\[
\vec{x} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} -2 - (-1) \\ 0 - 2 \\ 0 - 2 \end{pmatrix} = \begin{pmatrix} -1 - t \\ 2 - 2t \\ 2 - 2t \end{pmatrix}.
\]

(iii) Where do the line \( \ell \) and the plane \( \mathcal{P} \) intersect?

**Solution.** Any point on \( \ell \) is given by the parametrization from part (ii). Such a point will lie on \( \mathcal{P} \) if its coordinates satisfy the equation from part (i). Thus we want

\[
0 = 12x_1 - x_3 = 12(-1 - t) - (2 - 2t) = -14 - 10t \implies t = -\frac{7}{5}.
\]

2

Consider the points \( A(3,9,1) \), \( B(1,-2,391) \), and \( C(391,0,-1) \).

(i) Find the midpoint \( M \) of the line segment \( BC \).

**Solution.** The midpoint has position vector \( \vec{m} = \frac{1}{2}(\vec{c} + \vec{b}) \). Thus

\[
\vec{m} = \frac{1}{2} \begin{pmatrix} 391 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -2 \\ 391 \end{pmatrix} = \begin{pmatrix} 196 \\ -1 \\ 195 \end{pmatrix}.
\]
(ii) Find the area of the triangle $ABM$.

**Solution.** The area is $\frac{1}{2} ||\overrightarrow{AB} \times \overrightarrow{BM}||$. We compute the cross product:

$$\overrightarrow{AB} \times \overrightarrow{BM} = (\vec{b} - \vec{a}) \times (\vec{m} - \vec{b}) = \left(\begin{array}{c} -2 \\ -11 \\ 390 \end{array}\right) \times \left(\begin{array}{c} 195 \\ 1 \\ -196 \end{array}\right) = \left(\begin{array}{c} 1766 \\ -75758 \\ 2043 \end{array}\right).$$

Hence

$$\text{area } ABM = \frac{1}{2} \sqrt{(1766)^2 + (75758)^2 + (2043)^2} = \cdots$$

(iii) Find an equation for the plane $P$ through $A, B, C$.

**Solution.** We need a normal to the plane through $A, B, C$. There are many such normals and the default normal one computes would be $\overrightarrow{AB} \times \overrightarrow{AC}$ (or $\overrightarrow{AB} \times \overrightarrow{BC}$). However, $A, B, M$ also lie on the plane through $A, B, C$. Therefore $\overrightarrow{AB}$ and $\overrightarrow{BM}$ both are in this plane and their cross product is perpendicular to this plane. We can therefore use $\vec{n} = \overrightarrow{AB} \times \overrightarrow{BM}$, which we have already computed, as normal. So the equation is

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{a},$$

i.e. $1776x_1 - 75758x_2 + 2043x_3 = 1776 \cdot 391 - 75758 \cdot 9 + 2043 \cdot (-1) = 692373$.

(iv) What is the distance from the origin to the plane $P$ you just found?

**Solution.** The distance from any point $\vec{X}$ with position vector $\vec{x}$ to the plane $P$ is given by

$$d(\vec{X}, P) = \pm \frac{\vec{n} \cdot (\vec{x} - \vec{a})}{||\vec{n}||}$$

(you can replace $\vec{a}$ by the position vector of any other point on $P$.) If $\vec{X}$ is the origin, then $\vec{x} = \vec{0}$ and we get

$$d(O, P) = \frac{\vec{n} \cdot \vec{a}}{||\vec{n}||} = \pm \frac{1776 \cdot 3 - 75758 \cdot 9 + 2043 \cdot 1}{\sqrt{(1766)^2 + (75758)^2 + (2043)^2}} = \cdots$$

Consider the parametric curve $\vec{x}(t) = \left(\begin{array}{c} t - \sin t \\ \cos t \end{array}\right)$.

(i) Where does the tangent line at the point with $t = \frac{931}{3}\pi$ intersect the $x_2$ axis?

**Solution.** The tangent vector at $\vec{x}(t)$ is

$$\vec{x}'(t) = \left(\begin{array}{c} 1 - \cos t \\ -\sin t \end{array}\right).$$

At $t = \frac{391}{3}\pi$ one has $\vec{x}(t) = \left(\begin{array}{c} 391\pi/3 + \sqrt{3}/2 \\ -1/2 \end{array}\right)$ and $\vec{x}'(t) = \left(\begin{array}{c} 3/2 \\ -\sqrt{3}/2 \end{array}\right)$. The parametrization of the tangent line is therefore

$$\vec{x} = \left(\begin{array}{c} 391\pi/3 + \sqrt{3}/2 \\ -1/2 \end{array}\right) + s \left(\begin{array}{c} 3/2 \\ -\sqrt{3}/2 \end{array}\right) = \left(\begin{array}{c} 391\pi/3 + \sqrt{3}/2 + 3s/2 \\ -1/2 - s\sqrt{3}/2 \end{array}\right)$$

This line hits the $x_2$ axis when $391\pi/3 + \sqrt{3}/2 + 3s/2 = 0$, i.e. when

$$s = -\frac{782}{6}\pi - \frac{1}{3}\sqrt{3}.$$
(ii) For some value of $t$ with $0 < t < 2\pi$ the tangent to the parametric curve $\vec{x}(t) = \begin{pmatrix} t - \sin t \\ \cos t \end{pmatrix}$ is horizontal (i.e. parallel to the $x_1$ axis). What is $t$?

**Solution.** When is the vector $\vec{x}'(t)$ horizontal? This happens when its second component vanishes. Since

$$\vec{x}'(t) = \begin{pmatrix} 1 - \cos t \\ -\sin t \end{pmatrix}$$

this means that $\sin t = 0$. The only $t$ between 0 and $2\pi$ for which $\sin t = 0$ is $t = \pi$. 