

QUESTIONS FROM SOME OLD SECOND MIDTERMS

MATH 222, LECTURE 3, SPRING 2008

(1) Compute $\lim_{x \rightarrow 0} \frac{e^{x^2} \cos x - 1 - \frac{1}{2}x^2}{1 - \cos(x^2)}$

[Hint: For partial credit first compute the Taylor expansions up to $o(x^4)$ of $e^{x^2} \cos x$ and of $\cos(x^2)$.]

Solution. Follow the hint and compute:

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + o(x^4) \quad (\text{substitute } t = x^2 \text{ in } e^t)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$e^{x^2} \cos x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) \quad (\text{multiply})$$

$$e^{x^2} \cos x - 1 - \frac{1}{2}x^2 = \frac{1}{24}x^4 + o(x^4)$$

$$\cos x^2 = 1 - \frac{1}{2}x^4 + o(x^4) \quad (\text{put } t = x^2 \text{ in } \cos t)$$

$$1 - \cos x^2 = \frac{1}{2}x^4 + o(x^4)$$

So you get

$$\frac{e^{x^2} \cos x - 1 - \frac{1}{2}x^2}{1 - \cos(x^2)} = \frac{\frac{1}{24}x^4 + o(x^4)}{\frac{1}{2}x^4 + o(x^4)} = \frac{\frac{1}{24} + \frac{o(x^4)}{x^4}}{\frac{1}{2} + \frac{o(x^4)}{x^4}}$$

By definition of “little-oh” you have

$$\lim_{x \rightarrow 0} \frac{o(x^4)}{x^4} = 0$$

so that

$$\lim_{x \rightarrow 0} \frac{e^{x^2} \cos x - 1 - \frac{1}{2}x^2}{1 - \cos(x^2)} = \frac{1/24}{1/2} = \frac{1}{12}$$

□

(2) Show that the Taylor series for

$$f(x) = \sin(3x + 2)$$

converges for *all* real numbers x .

Solution. We must show that the remainder term in

$$f(x) = T_n f(x) + R_n f(x)$$

goes to zero as $n \rightarrow \infty$. Lagrange’s formula tells us that

$$R_n f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}$$

for some number c between 0 and x .

After differentiating $n + 1$ times you find that

$$f^{(n+1)}(x) = \pm 3^{n+1} \sin(3x + 2) \quad \text{or} \quad \pm 3^{n+1} \cos(3x + 2)$$

depending on whether n is even or odd.

In both cases you have

$$\left| f^{(n+1)}(x) \right| \leq 3^{n+1}$$

and therefore, no matter what c is, you get

$$|R_n f(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} \right| \leq \frac{3^{n+1}}{(n+1)!}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{3^{n+1}}{(n+1)!} = 0$$

the Sandwich Theorem implies that $\lim_{n \rightarrow \infty} R_n f(x) = 0$ and we are done. \square

(3) Find

$$\mathbf{Re} \left\{ \frac{e^{(1+i)x}}{2+i} \right\} \quad \text{and} \quad \mathbf{Re} \left\{ \frac{2+i}{e^{(1+i)x}} \right\}$$

Solution.

$$e^{(1+i)x} = e^{x+ix} = e^x e^{ix} = e^x (\cos x + i \sin x)$$

so

$$\begin{aligned} \frac{e^{(1+i)x}}{2+i} &= \frac{e^{(1+i)x}}{2+i} \times \frac{2-i}{2-i} \\ &= e^x \frac{\cos x + i \sin x}{2+i} \times \frac{2-i}{2-i} \\ &= e^x \frac{2 \cos x + \sin x + i(\dots)}{2^2 + 1^2} \\ &= \frac{1}{5} e^x (2 \cos x + \sin x + i(\dots)) \end{aligned}$$

The real part is therefore

$$\mathbf{Re} \left\{ \frac{e^{(1+i)x}}{2+i} \right\} = \frac{1}{5} e^x (2 \cos x + \sin x).$$

To compute $(2+i)/e^{(1+i)x}$ note that

$$\frac{1}{e^{(1+i)x}} = e^{-(1+i)x} = e^{-x} (\cos x - i \sin x).$$

Therefore

$$\begin{aligned} \frac{2+i}{e^{(1+i)x}} &= (2+i)e^{-x} (\cos x - i \sin x) \\ &= e^{-x} (2 \cos x + \sin x + i(\dots)) \end{aligned}$$

and the real part is

$$\mathbf{Re} \left\{ \frac{2+i}{e^{(1+i)x}} \right\} = e^{-x} (2 \cos x + \sin x).$$

Note that since the problem only asked for the real part of $e^{(1+i)x}/(2+i)$ there was no point in computing the imaginary terms “ $i(\dots)$ ” above. \square

(4) You are given an angle x whose Cosine and Sine are given by

$$\cos x = A, \quad \text{and} \quad \sin x = \sqrt{1 - A^2}.$$

Compute $\cos 5x$.

Solution. This problem asks for the “quintuple angel formula for the cosine.”
DeMoivre’s formula says

$$\cos 5x + i \sin 5x = e^{5ix} = (e^{ix})^5 = (\cos x + i \sin x)^5.$$

To expand the 5th power use

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

(fifth line in Pascal’s triangle). You get

$$(\cos x + i \sin x)^5 = (\cos x)^5 - 10(\cos x)^3(\sin x)^2 + 5(\cos x)(\sin x)^4 + i(\dots).$$

Taking the real part you get

$$\cos 5x = \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x.$$

Finally we use $\cos x = A$ and $\sin x = \sqrt{1 - A^2}$ to get

$$\cos 5x = A^5 - 10A^3(1 - A^2) + 5A(1 - A^2)^2.$$

□

- (5) (a) Compute the Taylor expansion up to $o(x^7)$ of $f(x) = \ln(1 + x^3) - x \sin x^2$.

Solution.

$$\begin{aligned} \ln(1 + x^3) &= x^3 - \frac{1}{2}x^6 + o(x^7) && \text{(substitute } t = x^3 \text{ in } \ln(1 + t)) \\ x \sin x^2 &= x^3 - \frac{1}{6}x^7 + o(x^7) = x^3 + o(x^7) && \text{(set } t = x^2 \text{ in } \sin t \text{ and multiply)} \\ f(x) &= -\frac{1}{2}x^6 + o(x^7) && \text{(subtract)} \end{aligned}$$

□

- (b) Compute the following limits

$$A = \lim_{x \rightarrow 0} \frac{f(x)}{x^5}, \quad B = \lim_{x \rightarrow 0} \frac{f(x)}{x^6}, \quad C = \lim_{x \rightarrow 0} \frac{f(x)}{x^7}.$$

where $f(x)$ is as above.

Solution. From part (a) we find

$$\begin{aligned} \frac{f(x)}{x^5} &= -\frac{1}{2}x + o(x^2) \\ \frac{f(x)}{x^6} &= -\frac{1}{2} + o(x) \\ \frac{f(x)}{x^7} &= -\frac{1}{2x} + o(1) \end{aligned}$$

Therefore $A = 0$, $B = -\frac{1}{2}$ and C does not exist. □

- (6) Show that the Taylor series for

$$f(x) = e^{-2x^4}$$

converges to $f(x)$ for *all* real numbers x .

Solution. Computing the derivatives $f^{(n)}(x)$ quickly becomes very difficult. Instead we do a substitution.

First, apply Lagrange's formula for the remainder to the function $g(t) = e^t$:

$$e^t = 1 + t + \cdots + \frac{t^n}{n!} + \frac{g^{(n+1)}(c)t^{n+1}}{(n+1)!} = 1 + t + \cdots + \frac{t^n}{n!} + \frac{e^c t^{n+1}}{(n+1)!}$$

for some number c between 0 and t . Now set $t = -2x^4$:

$$e^{-2x^4} = 1 - 2x^4 + \frac{2^2 x^8}{2!} - \cdots + \frac{(-2)^n x^{4n}}{n!} + \frac{e^c}{(n+1)!} (-2)^{n+1} x^{4n+4},$$

where c is a number between 0 and $t = -2x^4$.

The first part

$$1 - 2x^4 + \frac{2^2 x^8}{2!} - \cdots + \frac{(-2)^n x^{4n}}{n!}$$

is the Taylor polynomial of degree $4n$ of the function $f(x)$, and the last term

$$\frac{e^c}{(n+1)!} (-2)^{n+1} x^{4n+4},$$

is the remainder term, of which we have to show that it vanishes as $n \rightarrow \infty$.

Note that $t = -2x^4 < 0$, so c is a number between 0 and a negative number, and therefore is itself also negative. Hence $e^c \leq 1$. We therefore have

$$|\text{Remainder term}| = \frac{e^c}{(n+1)!} 2^{n+1} |x|^{4n+4} \leq \frac{2^{n+1} |x|^{4n+4}}{(n+1)!} = \frac{(2x^4)^{n+1}}{(n+1)!}$$

As $n \rightarrow \infty$ one has $\frac{(2x^4)^{n+1}}{(n+1)!} \rightarrow 0$ so the Sandwich theorem implies that the remainder term also goes to zero.

□

(7) Consider the Taylor series

$$f(x) = \frac{1+x}{(2+x)(3-x)} = f_0 + f_1x + f_2x^2 + \cdots + f_nx^n + \cdots$$

If $f_{4931} = A$ and $f_{4932} = B$ then compute f_{4933} and f_{4934} .

Solution. One has

$$f(x) = \frac{1+x}{x^2+x-6} = f_0 + f_1x + f_2x^2 + \cdots + f_nx^n + \cdots$$

Multiply both sides with $x^2 + x - 6$ and compare coefficients. You get

$$\begin{aligned} 1+x &= -6f_0 && \implies -6f_0 = 1 \\ &+ (-6f_1 + f_0)x && \implies -6f_1 + f_0 = 1 \\ &+ (-6f_2 + f_1 + f_0)x^2 && \implies -6f_2 + f_1 + f_0 = 0 \\ &+ (-6f_3 + f_2 + f_1)x^3 && \implies -6f_3 + f_2 + f_1 = 0 \\ &\vdots && \\ &+ (-6f_n + f_{n-1} + f_{n-2})x^n && \implies -6f_n + f_{n-1} + f_{n-2} = 0 \\ &+ o(x^n) && \end{aligned}$$

So for all $n \geq 2$ the coefficients f_{\dots} satisfy

$$-6f_n + f_{n-1} + f_{n-2} = 0$$

To get f_{4933} set $n = 4933$:

$$-6f_{4933} + f_{4932} + f_{4931} = 0 \implies f_{4933} = \frac{f_{4932} + f_{4931}}{6} = \frac{A+B}{6}.$$

To get f_{4930} choose $n = 4932$:

$$-6f_{4932} + f_{4931} + f_{4930} = 0 \implies f_{4930} = 6f_{4932} - f_{4931} = 6B - A.$$

□

(8) Let $z = 1 + i\sqrt{3}$.

(a) Find the smallest integer $n > 0$ for which z^n is a negative real number.

Solution. A complex number is negative and real if its argument is π (up to a multiple of 2π).

Make a drawing and you find that $\arg z = \pi/3$. Therefore $\arg z^2 = 2\pi/3$ and $\arg z^3 = \pi$. It follows that z^3 is the first power of z which is a negative real number. □

(b) Find the smallest integer $n > 0$ for which z^n is a positive real number.

Solution. A complex number is positive and real if its argument is a multiple of 2π .

Since $\arg z = \pi/3$ the first power of z which is positive real is z^6 .

□

(c) Same as above if $z = -\sqrt{3} + i$.

Solution. $\arg(-\sqrt{3} + i) = \frac{5}{6}\pi$, so $(-\sqrt{3} + i)^6$ is negative real, and $(-\sqrt{3} + i)^{12}$ is positive real. □

(d) Same as above if $z = -2 - 2i$.

Solution. $\arg(-2 - 2i) = \frac{5}{4}\pi$, so $(-2 - 2i)^4$ is negative real and $(-2 - 2i)^8$ is positive real. □

(e) Find all complex numbers z for which $z^2 = -i$.

Solution. If $z^2 = -i$ then $|z|^2 = |-i| = 1$ so $|z| = 1$.

Moreover, $2 \arg z = \arg z^2 = \arg(-i) = \frac{3}{2}\pi + 2k\pi$ (k an integer); in other words, $2 \arg z$ is $\frac{3}{2}\pi$ plus a multiple of 2π .

Divide by 2, and you find that $\arg z = \frac{3}{4}\pi + k\pi$, i.e. $\arg z$ is $\frac{3}{4}\pi$ plus a multiple of π .

This leads to two possibilities for z :

$$|z| = 1, \arg z = \frac{3}{4}\pi \implies z = -\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2},$$

and

$$|z| = 1, \arg z = \frac{3}{4}\pi + \pi \implies z = \frac{1}{2}\sqrt{2} - \frac{i}{2}\sqrt{2},$$

□

(f) Find all complex numbers z for which $z^3 = i$.

Solution. $|z|^3 = |z^3| = |i| = 1 \implies |z| = 1$.

$3 \arg z = \arg z^3 = \arg i = \frac{\pi}{2} + 2k\pi$ (k an integer). Hence $\arg z = \frac{\pi}{6} + k\frac{2\pi}{3}$, i.e. $\arg z$ is $\frac{\pi}{6}$ plus a multiple of $\frac{2\pi}{3}$ (120°).

This leads to three possibilities

$$\begin{aligned} |z| = 1, \arg z = \frac{\pi}{6} &\implies z = \frac{1}{2}\sqrt{3} + \frac{1}{2}i \\ |z| = 1, \arg z = \frac{1}{6}\pi + \frac{2}{3}\pi = \frac{5}{6}\pi &\implies z = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i \\ |z| = 1, \arg z = \frac{1}{6}\pi + \frac{4}{3}\pi = \frac{3}{2}\pi &\implies z = -i. \end{aligned}$$

□

(g) Find all complex numbers z for which $z^2 = 1 + i$.

Solution. $|z|^2 = |z|^2 = |1 + i| = \sqrt{2} \implies |z| = 2^{1/4}$
 $2 \arg z = \arg z^2 = \arg(1 + i) = \frac{1}{4}\pi + 2k\pi$, so $\arg z = \frac{1}{8}\pi + k\pi$. This leads to two solutions

$$|z| = 2^{1/4}, \arg z = \frac{1}{8}\pi \implies z = 2^{1/4} \cos \frac{\pi}{8} + i2^{1/4} \sin \frac{\pi}{8}$$

$$|z| = 2^{1/4}, \arg z = \frac{1}{8}\pi + \pi \implies z = -2^{1/4} \cos \frac{\pi}{8} - i2^{1/4} \sin \frac{\pi}{8}$$

Since $\pi/8$ is not one of the “familiar angles” we don’t simplify $\sin \pi/8$ and $\cos \pi/8$ (even though you could do this using the half-angle formulas). \square