

Problems on partial derivatives and the chain rule

- 1 Consider a function of two variables given by $u(x, y) = f(r(x, y))$ where $r(x, y) = \sqrt{x^2 + y^2}$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

- 2 (i) Consider a function of three variables given by $u(x, y, z) = f(\rho(x, y, z))$ where $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(\rho) + \frac{2}{\rho} f'(\rho)$$

- (ii) For which value(s) of a does the function $u = \rho^a$ satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0?$$

- 3 For which values of a, b is the function

$$u(x, t) = e^{at} \sin bx$$

a solution of the so-called *telegraph equation*

$$\frac{\partial^2 u}{\partial t^2} + K \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}?$$

Here K and c are positive constants.

- 4 Let $f(\xi)$ and $g(\eta)$ be two *arbitrary* functions of one variable. Show that the function of two variables given by

$$u(x, t) = f(x + ct) + g(x - ct)$$

satisfies the *one-dimensional wave equation*

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

- 5 If $f(\xi)$ is an arbitrary function of one variable, then find all values of c such that $u(x, t) = f(x - ct)$ satisfies

$$u_{tt} + 3u_{xt} - 2u_{xx} = 0.$$

- 6 For which values of the constants $k, l, m > 0$ is the function

$$u(x, y, t) = \sin(kt) \sin(lx) \sin(my)$$

a solution of the *two-dimensional wave equation*

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}?$$

(c is a constant.)