Vectors

40. Introduction to vectors

Definition 40.1. *A vector is a column of two, three, or more numbers, written as*

$$\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \textit{or} \quad \vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \textit{or} \quad \vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

in general.

*The **length of a vector** $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right)$ is defined by*

$$\|\vec{\boldsymbol{a}}\| = \left\| \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

We will always deal with either the two or three dimensional cases, in other words, the cases $n = 2$ or $n = 3$, respectively. For these cases there is a geometric description of vectors which is very useful. In fact, the two and three dimensional theories have their origins in mechanics and geometry. In higher dimensions the geometric description fails, simply because we cannot visualize a four dimensional space, let alone a higher dimensional space. Instead of a geometric description of vectors there is an abstract theory called ***Linear Algebra*** which deals with "vector spaces" of any dimension (even infinite!). This theory of vectors in higher dimensional spaces is very useful in science, engineering and economics. You can learn about it in courses like MATH 320 or 340/341.

40.1. *Basic arithmetic of vectors*

You can add and subtract vectors, and you can multiply them with arbitrary real numbers. this section tells you how.

The ***sum of two vectors*** is defined by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}, \tag{48}$$

and

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}.$$

The ***zero vector*** is defined by

$$\vec{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \textit{or} \quad \vec{\mathbf{0}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It has the property that

$$\vec{\boldsymbol{a}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}$$

no matter what the vector $\vec{\boldsymbol{a}}$ is.

You can multiply a vector $\vec{\boldsymbol{a}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with a real number t according to the rule

$$t\vec{\boldsymbol{a}} = \begin{pmatrix} ta_1 \\ ta_2 \\ ta_3 \end{pmatrix}.$$

In particular, "minus a vector" is defined by

$$-\vec{\boldsymbol{a}} = (-1)\vec{\boldsymbol{a}} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}.$$

The difference of two vectors is defined by

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}).$$

So, to subtract two vectors you subtract their components,

$$\vec{a} - \vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$$

◀ **40.2 Some GOOD examples.**

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ \pi \end{pmatrix} = \begin{pmatrix} -1 \\ 3+\pi \end{pmatrix} \qquad 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 12 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 \\ -12 \\ 3-\sqrt{2} \end{pmatrix} \qquad a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$0 \cdot \begin{pmatrix} 12\sqrt{39} \\ \pi^2 - \ln 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} \qquad \begin{pmatrix} t+t^2 \\ 1-t^2 \end{pmatrix} = (1+t)\begin{pmatrix} t \\ 1-t \end{pmatrix}$$

▶

◀ **40.3 Two very, very BAD examples.** Vectors must have the same size to be added, therefore

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \textbf{undefined!!!}$$

Vectors and numbers are different things, so an equation like

$$\vec{a} = 3 \quad \text{is nonsense!}$$

This equation says that some vector ($\vec{a}$) is equal to some number (in this case: 3). ***Vectors and numbers are never equal!*** ▶

40.2. *Algebraic properties of vector addition and multiplication*

Addition of vectors and multiplication of numbers and vectors were defined in such a way that the following always hold for any vectors $\vec{a}, \vec{b}, \vec{c}$ (of the same size) and any real numbers s, t

(49) $\quad \vec{a} + \vec{b} = \vec{b} + \vec{a}$ $\quad$ [vector addition is commutative]

(50) $\quad \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ $\quad$ [vector addition is associative]

(51) $\quad t(\vec{a} + \vec{b}) = t\vec{a} + t\vec{b}$ $\quad$ [first distributive property]

(52) $\quad (s+t)\vec{a} = s\vec{a} + t\vec{a}$ $\quad$ [second distributive property]

◀ **40.4 Prove (49).** Let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be two vectors, and consider both possible ways of adding them:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1+a_1 \\ b_2+a_2 \\ b_3+a_3 \end{pmatrix}$$

We know (or we have assumed long ago) that addition of real numbers is commutative, so that $a_1 + b_1 = b_1 + a_1$, etc. Therefore

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} = \begin{pmatrix} b_1+a_1 \\ b_2+a_2 \\ b_3+a_3 \end{pmatrix} = \vec{b} + \vec{a}.$$

This proves (49). ▶

◀ **40.5 Example.** If $\vec{v}$ and $\vec{w}$ are two vectors, we define

$$\vec{a} = 2\vec{v} + 3\vec{w}, \quad \vec{b} = -\vec{v} + \vec{w}.$$

Problem: Compute $\vec{a} + \vec{b}$ and $2\vec{a} - 3\vec{b}$ in terms of $\vec{v}$ and $\vec{w}$.

Solution:

$$\vec{a} + \vec{b} = (2\vec{v} + 3\vec{w}) + (-\vec{v} + \vec{w}) = (2-1)\vec{v} + (3+1)\vec{w} = \vec{v} + 4\vec{w}$$
$$2\vec{a} - 3\vec{b} = 2(2\vec{v} + 3\vec{w}) - 3(-\vec{v} + \vec{w}) = 4\vec{w} + 6\vec{w} + 3\vec{v} - 3\vec{w} = 7\vec{v} + 3\vec{w}.$$

Problem: Find s, t so that $s\vec{a} + t\vec{b} = \vec{v}$.

Solution: Simplifying $s\vec{a} + t\vec{b}$ you find

$$s\vec{a} + t\vec{b} = s(2\vec{v} + 3\vec{w}) + t(-\vec{v} + \vec{w}) = (2s - t)\vec{v} + (3s + t)\vec{w}.$$

One way to ensure that $s\vec{a} + t\vec{b} = \vec{v}$ holds is therefore to choose s and t to be the solutions of

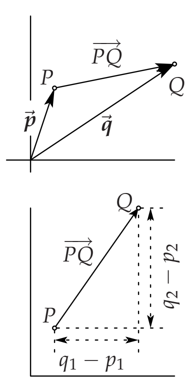
$$2s - t = 1$$
$$3s + t = 0$$

The second equation says $t = -3s$. The first equation then leads to $2s + 3s = 1$, i.e. $s = \frac{1}{5}$. Since $t = -3s$ we get $t = -\frac{3}{5}$. The solution we have found is therefore

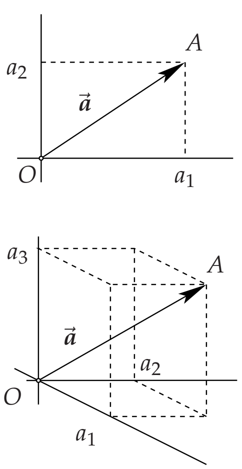
$$\tfrac{1}{5}\vec{a} - \tfrac{3}{5}\vec{b} = \vec{v}.$$

▶

40.3. *Geometric description of vectors*



two pictures of the vector $\overrightarrow{PQ} = \vec{q} - \vec{p}$



position vectors in the plane and in space

Vectors originally appeared in mechanics, where they represented forces: a force acting on some object has a ***magnitude*** and a ***direction***. Thus a force can be thought of as an arrow, where the length of the arrow indicates how strong the force is (how hard it pushes or pulls).

So we will think of vectors as ***arrows***: if you specify two points P and Q, then the arrow pointing from P to Q is a vector and we denote this vector by $\overrightarrow{PQ}$.

The precise mathematical definition is as follows:

Definition 40.6. *For any pair of points P and Q whose coordinates are (p_1, p_2, p_3) and (q_1, q_2, q_3) one defines a vector $\overrightarrow{PQ}$ by*

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}.$$

If the initial point of an arrow is the origin O, and the final point is any point Q, then the vector $\overrightarrow{OQ}$ is called the ***position vector*** *of the point Q.*

If $\vec{p}$ and $\vec{q}$ are the position vectors of P and Q, then one can write $\overrightarrow{PQ}$ as

$$\overrightarrow{PQ} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \vec{q} - \vec{p}.$$

For plane vectors we define $\overrightarrow{PQ}$ similarly, namely, $\overrightarrow{PQ} = \left(\begin{smallmatrix} q_1 - p_1 \\ q_2 - p_2 \end{smallmatrix}\right)$. The old formula for the distance between two points P and Q in the plane

$$\text{distance from } P \text{ to } Q = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

says that the length of the vector $\overrightarrow{PQ}$ is just the distance between the points P and Q, i.e.

$$\text{distance from } P \text{ to } Q = \left\|\overrightarrow{PQ}\right\|.$$

This formula is also valid if P and Q are points in space.

◀ **40.7 Example.** The point P has coordinates $(2, 3)$; the point Q has coordinates $(8, 6)$. The vector $\overrightarrow{PQ}$ is therefore

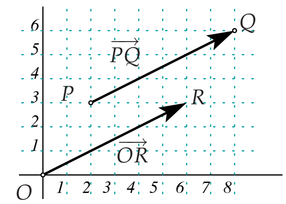
$$\overrightarrow{PQ} = \begin{pmatrix} 8 - 2 \\ 6 - 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

This vector is the position vector of the point R whose coordinates are $(6, 3)$. Thus

$$\overrightarrow{PQ} = \overrightarrow{OR} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

The distance from P to Q is the length of the vector $\overrightarrow{PQ}$, i.e.

$$\text{distance } P \text{ to } Q = \left\| \begin{pmatrix} 6 \\ 3 \end{pmatrix} \right\| = \sqrt{6^2 + 3^2} = 3\sqrt{5}.$$



▶

◀ **40.8 Example.** Find the distance between the points A and B whose position vectors are $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ respectively.

Solution: One has

$$\text{distance } A \text{ to } B = \|\overrightarrow{AB}\| = \|\vec{b} - \vec{a}\| = \left\| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\| = \sqrt{(-1)^2 + 0^2 + 1^2} = \sqrt{2}$$

▶

40.4. *Geometric interpretation of vector addition and multiplication*

Suppose you have two vectors $\vec{a}$ and $\vec{b}$. Consider them as position vectors, i.e. represent them by vectors that have the origin as initial point:

$$\vec{a} = \overrightarrow{OA}, \quad \vec{b} = \overrightarrow{OB}.$$

Then the origin and the three endpoints of the vectors $\vec{a}$, $\vec{b}$ and $\vec{a} + \vec{b}$ form a parallelogram. See figure 15.

To multiply a vector $\vec{a}$ with a real number t you multiply its length with $|t|$; if $t < 0$ you reverse the direction of $\vec{a}$.

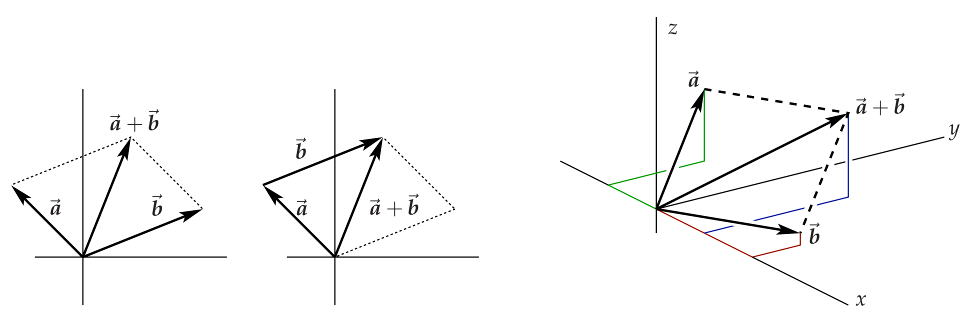


Figure 15. Two ways of adding plane vectors, and an addition of space vectors

◀ **40.9 Example.** In example 40.5 we assumed two vectors $\vec{v}$ and $\vec{w}$ were given, and then defined $\vec{a} = 2\vec{v} + 3\vec{w}$ and $\vec{b} = -\vec{v} + \vec{w}$. In figure 17 the vectors $\vec{a}$ and $\vec{b}$ are constructed geometrically from some arbitrarily chosen $\vec{v}$ and $\vec{w}$. We also found algebraically in example 40.5 that $\vec{a} + \vec{b} = \vec{v} + 4\vec{w}$. The third drawing in figure 17 illustrates this. ▶

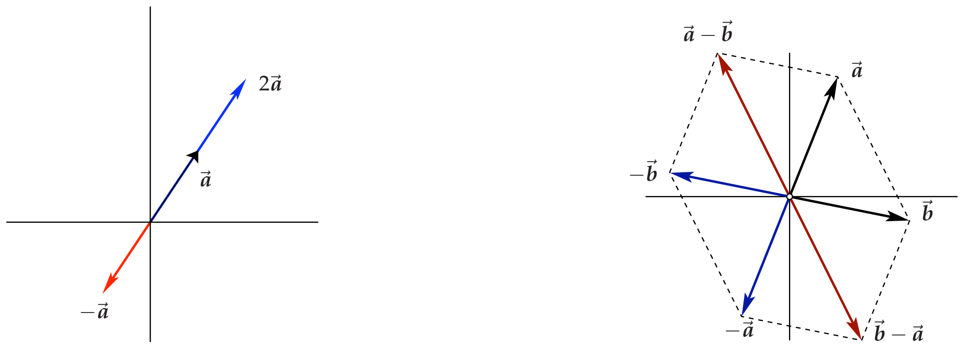


Figure 16. Multiples of a vector, and the difference of two vectors.

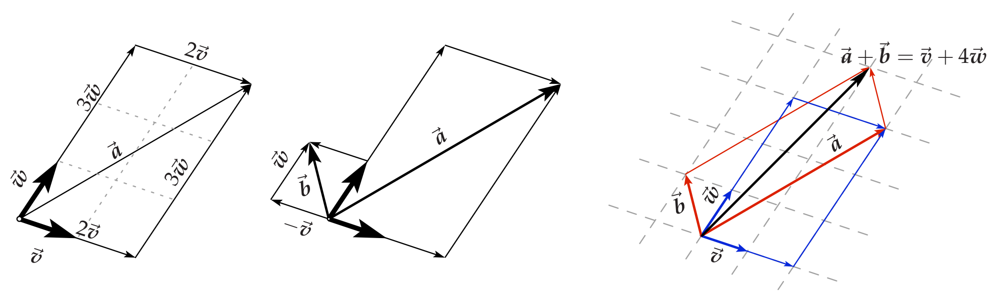
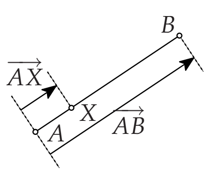


Figure 17. Picture proof that $\vec{a}+\vec{b}=\vec{v}+4\vec{w}$ in example 40.9.

41. Parametric equations for lines and planes



Given two ***distinct*** points A and B we consider the line segment AB. If X is any given point on AB then we will now find a formula for the position vector of X.

Define t to be the ratio between the lengths of the line segments AX and AB,

$$t = \frac{\text{length } AX}{\text{length } AB}.$$

Then the vectors $\overrightarrow{AX}$ and $\overrightarrow{AB}$ are related by $\overrightarrow{AX} = t\overrightarrow{AB}$. Since AX is shorter than AB we have $0 < t < 1$.

The position vector of the point X on the line segment AB is

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \overrightarrow{OA} + t\overrightarrow{AB}.$$

If we write $\vec{a}, \vec{b}, \vec{x}$ for the position vectors of A, B, X, then we get

$$\vec{x} = (1-t)\vec{a} + t\vec{b} = \vec{a} + t(\vec{b} - \vec{a}). \tag{53}$$

This equation is called the ***parametric equation for the line through*** A ***and*** B**.** In our derivation the parameter t satisfied $0 \le t \le 1$, but there is nothing that keeps us from substituting negative values of t, or numbers $t > 1$ in (53). The resulting vectors $\vec{x}$ are position vectors of points X which lie on the line ℓ through A and B.

◀ **41.1 Find the parametric equation for the line ℓ through the points $A(2,1)$ and $B(3,-1)$, and determine where ℓ intersects the x_1 axis.**

Solution: The position vectors of A, B are $\vec{a} = \binom{1}{2}$ and $\vec{b} = \binom{3}{-1}$, so the position vector of an arbitrary point on ℓ is given by

$$\vec{x} = \vec{a} + t(\vec{b} - \vec{a}) = \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}3-1\\-1-2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + t\begin{pmatrix}2\\-3\end{pmatrix} = \begin{pmatrix}1+2t\\2-3t\end{pmatrix}$$

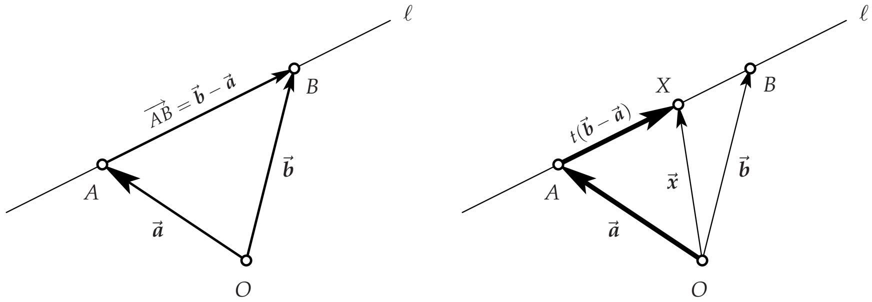
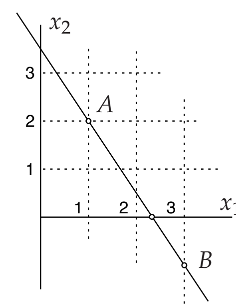


Figure 18. Constructing points on the line through A and B



where t is an arbitrary real number.

This vector points to the point $X = (1 + 2t, 2 - 3t)$. By definition, a point lies on the x_1-axis if its x_2 component vanishes. Thus if the point

$$X = (1 + 2t, 2 - 3t)$$

lies on the x_1-axis, then $2 - 3t = 0$, i.e. $t = \frac{2}{3}$. The intersection point X of ℓ and the x_1-axis is therefore $X|_{t=2/3} = (1 + 2 \cdot \frac{2}{3}, 0) = (\frac{5}{3}, 0)$.

▶

◀ **41.2 Midpoint of a line segment.** If M is the midpoint of the line segment AB, then the vectors $\overrightarrow{AM}$ and $\overrightarrow{MB}$ are both parallel and have the same direction and length (namely, half the length of the line segment AB). Hence they are equal: $\overrightarrow{AM} = \overrightarrow{MB}$. If $\vec{a}$, $\vec{m}$, and $\vec{b}$ are the position vectors of A , M and B, then this means

$$\vec{m} - \vec{a} = \overrightarrow{AM} = \overrightarrow{MB} = \vec{b} - \vec{m}.$$

Add $\vec{m}$ and $\vec{a}$ to both sides, and divide by 2 to get

$$\vec{m} = \tfrac{1}{2}\vec{a} + \tfrac{1}{2}\vec{b} = \frac{\vec{a} + \vec{b}}{2}.$$

▶

41.1. *Parametric equations for planes in space**

You can specify a plane in three dimensional space by naming a point A on the plane $\mathcal{P}$, and two vectors $\vec{v}$ and $\vec{w}$ parallel to the plane $\mathcal{P}$, but not parallel to each other. Then any point on the plane $\mathcal{P}$ has position vector $\vec{x}$ given by

$$\vec{x} = \vec{a} + s\vec{v} + t\vec{w}. \tag{54}$$

The following construction explains why (54) will give you any point on the plane through A, parallel to $\vec{v}, \vec{w}$.

Let A, $\vec{v}$, $\vec{w}$ be given, and suppose we want to express the position vector of some other point X on the plane $\mathcal{P}$ in terms of $\vec{a} = \overrightarrow{OA}$, $\vec{v}$, and $\vec{w}$.

First we note that

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}.$$

Next, you draw a parallelogram in the plane $\mathcal{P}$ whose sides are parallel to the vectors $\vec{v}$ and $\vec{w}$, and whose diagonal is the line segment AX. The sides of this parallelogram represent vectors which are multiples of $\vec{v}$ and $\vec{w}$ and which add up to $\overrightarrow{AX}$. So, if one side of the parallelogram is $s\vec{v}$ and the other $t\vec{w}$ then we have $\overrightarrow{AX} = s\vec{v} + t\vec{w}$. With $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$ this implies (54).

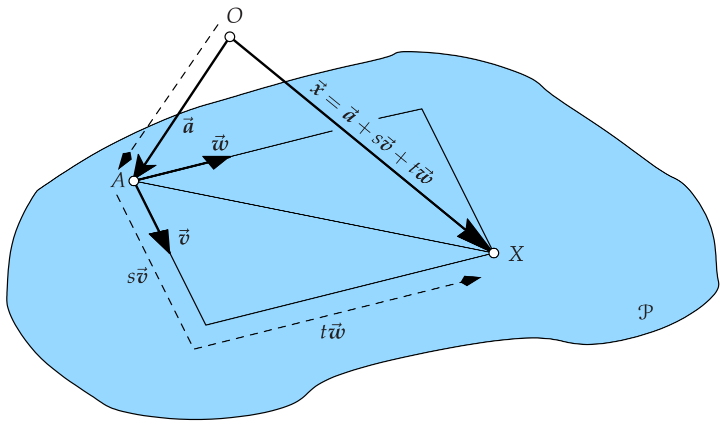


Figure 19. Generating points on a plane $\mathcal{P}$

42. Vector Bases

42.1. *The Standard Basis Vectors*

The notation for vectors which we have been using so far is not the most traditional. In the late 19th century GIBBS and HEAVYSIDE adapted HAMILTON's theory of Quaternions to deal with vectors. Their notation is still popular in texts on electromagnetism and fluid mechanics.

Define the following three vectors:

$$\vec{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \vec{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \vec{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Then *every vector can be written as a linear combination of $\vec{i}$, $\vec{j}$ and $\vec{k}$*, namely as follows:

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

Moreover, *there is only one way to write a given vector as a linear combination of $\{\vec{i}, \vec{j}, \vec{k}\}$*. This means that

$$a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} \iff \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}$$

For plane vectors one defines

$$\vec{i} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \vec{j} = \begin{pmatrix}0\\1\end{pmatrix}$$

and just as for three dimensional vectors one can write every (plane) vector $\vec{a}$ as a linear combination of $\vec{i}$ and $\vec{j}$,

$$\begin{pmatrix}a_1\\a_2\end{pmatrix} = a_1\vec{i} + a_2\vec{j}.$$

Just as for space vectors, there is only one way to write a given vector as a linear combination of $\vec{i}$ and $\vec{j}$.

42.2. *A Basis of Vectors (in general)**

The vectors $\vec{i}, \vec{j}, \vec{k}$ are called the ***standard basis vectors***. They are an example of what is called a "basis". Here is the definition in the case of space vectors:

Definition 42.1. *A triple of space vectors* $\{\vec{u}, \vec{v}, \vec{w}\}$ *is a* ***basis*** *if every space vector* $\vec{a}$ *can be written as a linear combination of* $\{\vec{u}, \vec{v}, \vec{w}\}$*, i.e.*

$$\vec{a} = a_u\vec{u} + a_v\vec{v} + a_w\vec{w},$$

and if there is only one way to do so for any given vector $\vec{a}$ *(i.e. the vector* $\vec{a}$ *determines the coefficients* a_u, a_v, a_w*).*

For plane vectors the definition of a basis is almost the same, except that a basis consists of two vectors rather than three:

Definition 42.2. *A pair of plane vectors* $\{\vec{u}, \vec{v}\}$ *is a* ***basis*** *if every plane vector* $\vec{a}$ *can be written as a linear combination of* $\{\vec{u}, \vec{v}\}$*, i.e.* $\vec{a} = a_u\vec{u} + a_v\vec{v}$*, and if there is only one way to do so for any given vector* $\vec{a}$ *(i.e. the vector* $\vec{a}$ *determines the coefficients* a_u, a_v*).*

43. Dot Product

Definition 43.1. *The "inner product" or "dot product" of two vectors is given by*

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3.$$

Note that the dot-product of two vectors is a number!

The dot product of two plane vectors is (predictably) defined by

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2.$$

An important property of the dot product is its relation with the length of a vector:

$$\|\vec{a}\|^2 = \vec{a} \bullet \vec{a}. \tag{55}$$

43.1. *Algebraic properties of the dot product*

The dot product satisfies the following rules,

$$\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a} \tag{56}$$

$$\vec{a} \bullet (\vec{b} + \vec{c}) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c} \tag{57}$$

$$(\vec{b} + \vec{c}) \bullet \vec{a} = \vec{b} \bullet \vec{a} + \vec{c} \bullet \vec{a} \tag{58}$$

$$t(\vec{a} \bullet \vec{b}) = (t\vec{a}) \bullet \vec{b} \tag{59}$$

which hold for all vectors $\vec{a}, \vec{b}, \vec{c}$ and any real number t.

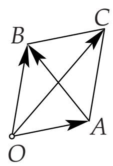
◀ **43.2 Example.** Simplify $\|\vec{a} + \vec{b}\|^2$.

One has

$$\begin{aligned} \|\vec{a} + \vec{b}\|^2 &= (\vec{a} + \vec{b}) \bullet (\vec{a} + \vec{b}) \\ &= \vec{a} \bullet (\vec{a} + \vec{b}) + \vec{b} \bullet (\vec{a} + \vec{b}) \\ &= \vec{a} \bullet \vec{a} + \underbrace{\vec{a} \bullet \vec{b} + \vec{b} \bullet \vec{a}}_{=2\vec{a}\bullet\vec{b} \text{ by (56)}} + \vec{b} \bullet \vec{b} \\ &= \|\vec{a}\|^2 + 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2 \end{aligned}$$

▶

43.2. *The diagonals of a parallelogram*



Here is an example of how you can use the algebra of the dot product to prove something in geometry.

Suppose you have a parallelogram one of whose vertices is the origin. Label the vertices, starting at the origin and going around counterclockwise, O, A, C and B. Let $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, $\vec{c} = \overrightarrow{OC}$. One has

$$\overrightarrow{OC} = \vec{c} = \vec{a} + \vec{b}, \quad \text{and} \quad \overrightarrow{AB} = \vec{b} - \vec{a}.$$

These vectors correspond to the diagonals OC and AB

Theorem 43.3. *In a parallelogram OACB the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the lengths of all four sides.*

Proof. The squared lengths of the diagonals are

$$\|\overrightarrow{OC}\|^2 = \|\vec{a} + \vec{b}\|^2 = \|\vec{a}\|^2 + 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2$$
$$\|\overrightarrow{AB}\|^2 = \|\vec{a} - \vec{b}\|^2 = \|\vec{a}\|^2 - 2\vec{a} \bullet \vec{b} + \|\vec{b}\|^2$$

Adding both these equations you get

$$\|\overrightarrow{OC}\|^2 + \|\overrightarrow{AB}\|^2 = 2\left(\|\vec{a}\|^2 + \|\vec{b}\|^2\right).$$

The squared lengths of the sides are

$$\|\overrightarrow{OA}\|^2 = \|\vec{a}\|^2, \quad \|\overrightarrow{AB}\|^2 = \|\vec{b}\|^2, \quad \|\overrightarrow{BC}\|^2 = \|\vec{a}\|^2, \quad \|\overrightarrow{OC}\|^2 = \|\vec{b}\|^2.$$

Together these also add up to $2\left(\|\vec{a}\|^2 + \|\vec{b}\|^2\right)$. □

43.3. *The dot product and the angle between two vectors*

Here is the most important interpretation of the dot product:

Theorem 43.4. *If the angle between two vectors $\vec{a}$ and $\vec{b}$ is θ, then one has*

$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta.$$

Proof. We need *the law of cosines* from high-school trigonometry. Recall that for a triangle OAB with angle θ at the point O, and with sides OA and OB of lengths a and b, the length c of the opposing side AB is given by

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{60}$$

In trigonometry this is proved by dropping a perpendicular line from B onto the side OA. The triangle

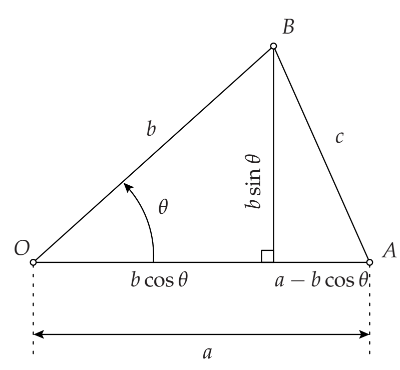
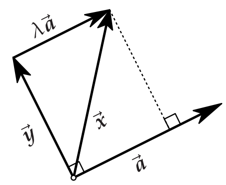


Figure 20. Proof of the law of cosines

OAB gets divided into two right triangles, one of which has AB as hypotenuse. Pythagoras then implies

$$c^2 = (b\sin\theta)^2 + (a - b\cos\theta)^2.$$



After simplification you get (60).

To prove the theorem you let O be the origin, and then observe that the length of the side AB is the length of the vector $\overrightarrow{AB} = \vec{b} - \vec{a}$. Here $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$, and hence

$$c^2 = \|\vec{b} - \vec{a}\|^2 = (\vec{b} - \vec{a}) \bullet (\vec{b} - \vec{a}) = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\vec{a} \bullet \vec{b}.$$

Compare this with (60), keeping in mind that $a = \|\vec{a}\|$ and $b = \|\vec{b}\|$: you are led to conclude that $-2\vec{a} \bullet \vec{b} = -2ab\cos\theta$, and thus $\vec{a} \bullet \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$. □

43.4. *Orthogonal projection of one vector onto another*

The following construction comes up very often. Let $\vec{a} \neq \vec{0}$ be a given vector. Then for any other vector $\vec{x}$ there is a number λ such that

$$\vec{x} = \lambda\vec{a} + \vec{y}$$

where $\vec{y} \perp \vec{a}$. In other words, you can write any vector $\vec{x}$ as the sum of one vector parallel to $\vec{a}$ and another vector orthogonal to $\vec{a}$. The two vectors $\lambda\vec{a}$ and $\vec{y}$ are called the ***parallel*** and ***orthogonal components*** of the vector $\vec{x}$ (with respect to $\vec{a}$), and sometimes the following notation is used

$$\vec{x}^{\parallel} = \lambda\vec{a}, \qquad \vec{x}^{\perp} = \vec{y}.,$$

so that

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}.$$

There are moderately simple formulas for $\vec{x}^{\parallel}$ and $\vec{x}^{\perp}$, but it is better to remember the following derivation of these formulas.

Assume that the vectors $\vec{a}$ and vx are given. Then we look for a number λ such that $\vec{y} = \vec{x} - \lambda\vec{a}$ is perpendicular to $\vec{a}$. Recall that $\vec{a} \perp (\vec{x} - \lambda\vec{a})$ if and only if

$$\vec{a} \bullet (\vec{x} - \lambda\vec{a}) = 0.$$

Expand the dot product and you get this equation for λ

$$\vec{a} \bullet \vec{x} - \lambda\vec{a} \bullet \vec{a} = 0,$$

whence

$$\lambda = \frac{\vec{a} \bullet \vec{x}}{\vec{a} \bullet \vec{a}} = \frac{\vec{a} \bullet \vec{x}}{\|\vec{a}\|^2} \tag{61}$$

To compute the parallel and orthogonal components of $\vec{x}$ w.r.t. $\vec{a}$ you first compute λ according to (61), which tells you that the parallel coponent is given by

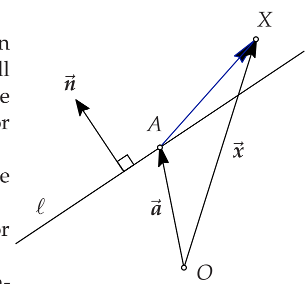
$$\vec{x}^{\parallel} = \lambda\vec{a} = \frac{\vec{a} \bullet \vec{x}}{\vec{a} \bullet \vec{a}}\,\vec{a}.$$

The orthogonal component is then "the rest," i.e. by definition $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel}$, so

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - \frac{\vec{a} \bullet \vec{x}}{\vec{a} \bullet \vec{a}}\,\vec{a}.$$

43.5. *Defining equations of lines*

In § 41 we saw how to generate points on a line given two points on that line by means of a "parametrization." I.e. given points A and B on the line ℓ the point whose position vector is $\vec{x} = \vec{a} + t(\vec{b} - \vec{a})$ will be on ℓ for any value of the "parameter" t.



Is X on ℓ?

In this section we will use the dot-product to give a different description of lines in the plane (and planes in three dimensional space.) We will derive an equation for a line. Rather than generating points on the line ℓ this equation tells us if any given point X in the plane is on the line or not.
Here is the derivation of the equation of a line in the plane. To produce the equation you need two ingredients:
1. One particular point on the line (let's call this point A, and write $\vec{a}$ for its position vector),
2. a ***normal vector*** $\vec{n}$ for the line, i.e. a nonzero vector which is perpendicular to the line.
Now let X be any point in the plane, and consider the line segment AX.

- Clearly, X will be on the line if and only if AX is parallel to ℓ [8]
- Since ℓ is perpendicular to $\vec{n}$, the segment AX and the line ℓ will be parallel if and only if $AX \perp \vec{n}$.
- $AX \perp \vec{n}$ holds if and only if $\overrightarrow{AX} \bullet \vec{n} = 0$.

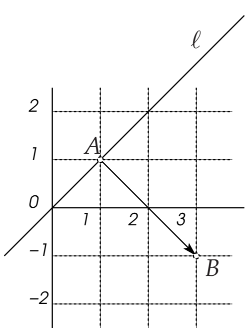
So in the end we see that X lies on the line ℓ if and only if the following vector equation is satisfied:

$$\overrightarrow{AX} \bullet \vec{n} = 0 \quad \text{or} \quad (\vec{x} - \vec{a}) \bullet \vec{n} = 0 \tag{62}$$

This equation is called a ***defining equation for the line*** ℓ**.**

Any given line has many defining equations. Just by changing the length of the normal you get a different equation, which still describes the same line.

◀ **43.5 Problem.** Find a defining equation for the line ℓ which goes through $A(1, 1)$ and is perpendicular to the line segment AB where B is the point $(3, -1)$.



Solution. We already know a point on the line, namely A, but we still need a normal vector. The line is required to be perpendicular to AB, so $\vec{n} = \overrightarrow{AB}$ is a normal vector:

$$\vec{n} = \overrightarrow{AB} = \begin{pmatrix} 3-1 \\ (-1)-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Of course any multiple of $\vec{n}$ is also a normal vector, for instance

$$\vec{m} = \tfrac{1}{2}\vec{n} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is a normal vector.

With $\vec{a} = \binom{1}{1}$ we then get the following equation for ℓ

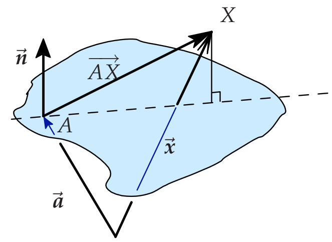
$$\vec{n} \bullet (\vec{x} - \vec{a}) = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix} = 2x_1 - 2x_2 = 0.$$

If you choose the normal $\vec{m}$ instead, you get

$$\vec{m} \bullet (\vec{x} - \vec{a}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \end{pmatrix} = x_1 - x_2 = 0.$$

Both equations $2x_1 - 2x_2 = 0$ and $x_1 - x_2 = 0$ are equivalent. ▶

[8] From plane Euclidean geometry: parallel lines either don't intersect or they coincide.



43.6. *Distance to a line*

Let ℓ be a line in the plane and assume a point A on the line as well as a vector $\vec{n}$ perpendicular to ℓ are known. Using the dot product one can easily compute the distance from the line to any other given point P in the plane. Here is how:

Draw the line m through A perpendicular to ℓ, and drop a perpendicular line from P onto m. let Q be the projection of P onto m. The distance from P to ℓ is then equal to the length of the line segment AQ. Since AQP is a right triangle one has

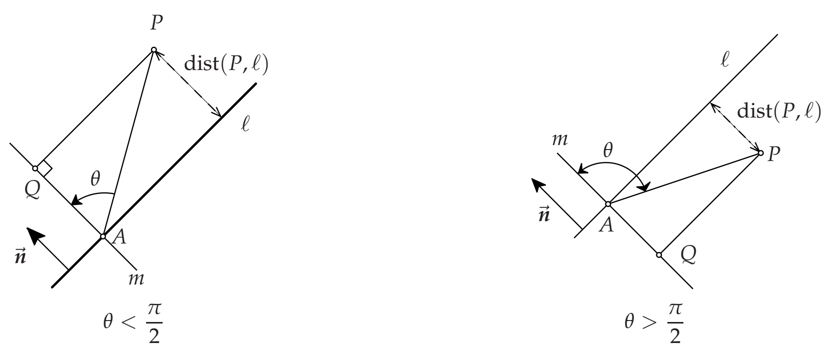
$$AQ = AP\cos\theta.$$

Here θ is the angle between the normal $\vec{n}$ and the vector $\overrightarrow{AP}$. One also has

$$\vec{n} \bullet (\vec{p} - \vec{a}) = \vec{n} \bullet \overrightarrow{AP} = \|\overrightarrow{AP}\|\ \|\vec{n}\|\cos\theta = AP\,\|\vec{n}\|\cos\theta.$$

Hence we get

$$\operatorname{dist}(P,\ell) = \frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|}.$$



This argument from a drawing contains a hidden assumption, namely that the point P lies on the side of the line ℓ pointed to by the vector $\vec{n}$. If this is not the case, so that $\vec{n}$ and $\overrightarrow{AP}$ point to opposite sides of ℓ, then the angle between them exceeds 90°, i.e. $\theta > \pi/2$. In this case $\cos\theta < 0$, and one has $AQ = -AP\cos\theta$. the distance formula therefore has to be modified to

$$\operatorname{dist}(P,\ell) = -\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|}.$$

43.7. *Defining equation of a plane*

Just as we have seen how you can form the defining equation for a line in the plane from just one point on the line and one normal vector to the line, you can also form the defining equation for a plane in space, again knowing only one point on the plane, and a vector perpendicular to it.

If A is a point on some plane $\mathcal{P}$ and $\vec{n}$ is a vector perpendicular to $\mathcal{P}$, then any other point X lies on $\mathcal{P}$ if and only if $\overrightarrow{AX} \perp \vec{n}$. In other words, in terms of the position vectors $\vec{a}$ and $\vec{x}$ of A and X,

$$\text{the point } X \text{ is on } \mathcal{P} \iff \vec{n} \bullet (\vec{x} - \vec{a}) = 0.$$

Arguing just as in § 43.6 you find that the distance of a point X in space to the plane $\mathcal{P}$ is

$$\operatorname{dist}(X, \mathcal{P}) = \pm \frac{\vec{n} \bullet (\vec{x} - \vec{a})}{\|\vec{n}\|}. \tag{63}$$

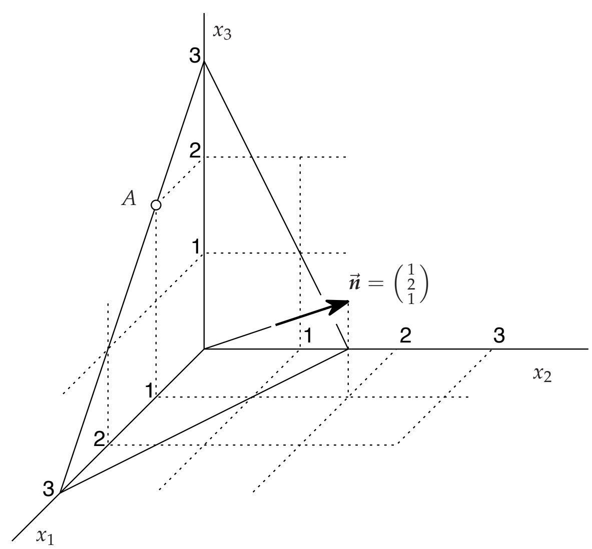
Here the sign is "+" if X and the normal $\vec{n}$ are on the same side of the plane $\mathcal{P}$; otherwise the sign is "−".

◀ 43.6 Find the defining equation for the plane $\mathcal{P}$ through the point $A(1,0,2)$ which is perpendicular to the vector $\begin{pmatrix}1\\2\\1\end{pmatrix}$.

Solution: We know a point (A) and a normal vector $\vec{n} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ for $\mathcal{P}$. Then any point X with coordinates (x_1, x_2, x_3), or, with position vector $\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$, will lie on the plane $\mathcal{P}$ if and only if

$$\begin{aligned}
\vec{n} \bullet (\vec{x} - \vec{a}) = 0 &\iff \begin{pmatrix}1\\2\\1\end{pmatrix} \bullet \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} \right\} = 0 \\
&\iff \begin{pmatrix}1\\2\\1\end{pmatrix} \bullet \begin{pmatrix}x_1 - 1\\x_2\\x_3 - 2\end{pmatrix} = 0 \\
&\iff 1 \cdot (x_1 - 1) + 2 \cdot (x_2) + 1 \cdot (x_3 - 2) = 0 \\
&\iff x_1 + 2x_2 + x_3 - 3 = 0.
\end{aligned}$$

▶



◀ 43.7 Let $\mathcal{P}$ be the plane from the previous example. Which of the points $P(0,0,1)$, $Q(0,0,2)$, $R(-1,2,0)$ and $S(-1,0,5)$ lie on $\mathcal{P}$? Compute the distances from the points P, Q, R, S to the plane $\mathcal{P}$. Separate the points which do not lie on $\mathcal{P}$ into two group of points which lie on the same side of $\mathcal{P}$.

Solution: We apply (63) to the position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ of the points P, Q, R, S. For each calculation we need

$$\|\vec{n}\| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}.$$

The third component of the given normal $\vec{n} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ is positive, so $\vec{n}$ points "upwards." Therefore, if a point lies on the side of $\mathcal{P}$ pointed to by $\vec{n}$, we shall say that the point lies *above the plane.*

P: $\vec{p} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-1) + 2 \cdot (0) + 1 \cdot (-1) = -2$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = -\frac{2}{\sqrt{6}} = -\frac{1}{3}\sqrt{6}.$$

This quantity is negative, so P lies below $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{3}\sqrt{6}$.

Q: $\vec{q} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-1) + 2 \cdot (0) + 1 \cdot (0) = -1$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = -\frac{1}{\sqrt{6}} = -\frac{1}{6}\sqrt{6}.$$

This quantity is negative, so Q also lies below $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{6}\sqrt{6}$.

R: $\vec{r} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-2) + 2 \cdot (2) + 1 \cdot (-2) = 0$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = 0.$$

Thus R lies on the plane $\mathcal{P}$, and its distance to $\mathcal{P}$ is of course 0.

S: $\vec{s} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$, $\vec{p} - \vec{a} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$, $\vec{n} \bullet (\vec{p} - \vec{a}) = 1 \cdot (-1) + 2 \cdot (0) + 1 \cdot (3) = 2$

$$\frac{\vec{n} \bullet (\vec{p} - \vec{a})}{\|\vec{n}\|} = \frac{2}{\sqrt{6}} = \frac{1}{3}\sqrt{6}.$$

This quantity is positive, so S lies above $\mathcal{P}$. Its distance to $\mathcal{P}$ is $\frac{1}{3}\sqrt{6}$.

We have found that P and Q lie below the plane, R lies on the plane, and S is above the plane. ▶

◀ 43.8 Where does the line through the points $B(2,0,0)$ and $C(0,1,2)$ intersect the plane $\mathcal{P}$ from example 43.6?

Solution: Let ℓ be the line through B and C. We set up the parametric equation for ℓ. According to §41, (53) every point X on ℓ has position vector $\vec{x}$ given by

$$\vec{x} = \vec{b} + t(\vec{c} - \vec{b}) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0-2 \\ 1-0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 2-2t \\ t \\ 2t \end{pmatrix} \tag{64}$$

for some value of t.

The point X whose position vector $\vec{x}$ is given above lies on the plane $\mathcal{P}$ if $\vec{x}$ satisfies the defining equation of the plane. In example 43.6 we found this defining equation. It was

$$\vec{n} \bullet (\vec{x} - \vec{a}) = 0, \text{ i.e. } x_1 + 2x_2 + x_3 - 3 = 0. \tag{65}$$

So to find the point of intersection of ℓ and $\mathcal{P}$ you substitute the parametrization (64) in the defining equation (65):

$$0 = x_1 + 2x_2 + x_3 - 3 = (2-2t) + 2(t) + (2t) - 3 = 2t - 1.$$

This implies $t = \frac{1}{2}$, and thus the intersection point has position vector

$$\vec{x} = \vec{b} + \tfrac{1}{2}(\vec{c} - \vec{b}) = \begin{pmatrix} 2-2t \\ t \\ 2t \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix},$$

i.e. ℓ and $\mathcal{P}$ intersect at $X(1, \frac{1}{2}, 1)$. ▶

44. Cross Product

44.1. *Algebraic definition of the cross product*

Here is the definition of the cross-product of two vectors. The definition looks a bit strange and arbitrary at first sight – it really makes you wonder who thought of this. We will just put up with that for now and explore the properties of the cross product. Later on we will see a geometric interpretation of the cross product which will show that this particular definition is really useful. We will also find a few tricks that will help you reproduce the formula without memorizing it.

$\times$	$\vec{\boldsymbol{i}}$	$\vec{\boldsymbol{j}}$	$\vec{\boldsymbol{k}}$
$\vec{\boldsymbol{i}}$	$\vec{\boldsymbol{0}}$	$\vec{\boldsymbol{k}}$	$-\vec{\boldsymbol{j}}$
$\vec{\boldsymbol{j}}$	$-\vec{\boldsymbol{k}}$	$\vec{\boldsymbol{0}}$	$\vec{\boldsymbol{i}}$
$\vec{\boldsymbol{k}}$	$\vec{\boldsymbol{j}}$	$-\vec{\boldsymbol{i}}$	$\vec{\boldsymbol{0}}$

Definition 44.1. *The "outer product" or "cross product" of two vectors is given by*

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Note that the cross-product of two vectors is again a vector!

◀ **44.2 Example.** If you set $\vec{\boldsymbol{b}} = \vec{\boldsymbol{a}}$ in the definition you find the following important fact: *The cross product of any vector with itself is the zero vector:*

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{a}} = \vec{\boldsymbol{0}} \quad \text{for } \textbf{\textit{any}} \text{ vector } \vec{\boldsymbol{a}}.$$

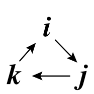
▶

◀ **44.3 Example.** Let $\vec{\boldsymbol{a}} = \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$, $\vec{\boldsymbol{b}} = \left(\begin{smallmatrix} -2 \\ 1 \\ 0 \end{smallmatrix}\right)$ and compute the cross product of these vectors.

Solution:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 - 3 \cdot 1 \\ 3 \cdot (-2) - 1 \cdot 0 \\ 1 \cdot 1 - 2 \cdot (-2) \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \\ 5 \end{pmatrix}$$

▶



In terms of the standard basis vectors you can check the *multiplication table*. An easy way to remember the multiplication table is to put the vectors $\vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}, \vec{\boldsymbol{k}}$ clockwise in a circle. Given two of the three vectors their product is either plus or minus the remaining vector. To determine the sign you step from the first vector to the second, to the third: if this makes you go clockwise you have a plus sign, if you have to go counterclockwise, you get a minus.

The products of $\vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}$ and $\vec{\boldsymbol{k}}$ are all you need to know to compute the cross product. Given two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ write them as $\vec{\boldsymbol{a}} = a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}} + a_3\vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{b}} = b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}}$, and multiply as follows

$$\begin{aligned}
\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} &= (a_1\vec{\boldsymbol{i}} + a_2\vec{\boldsymbol{j}} + a_3\vec{\boldsymbol{k}}) \times (b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}}) \\
&= \quad a_1\vec{\boldsymbol{i}} \times (b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}}) \\
&\quad + a_2\vec{\boldsymbol{j}} \times (b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}}) \\
&\quad + a_3\vec{\boldsymbol{k}} \times (b_1\vec{\boldsymbol{i}} + b_2\vec{\boldsymbol{j}} + b_3\vec{\boldsymbol{k}}) \\
&= a_1b_1\vec{\boldsymbol{i}} \times \vec{\boldsymbol{i}} + a_1b_2\vec{\boldsymbol{i}} \times \vec{\boldsymbol{j}} + a_1b_3\vec{\boldsymbol{i}} \times \vec{\boldsymbol{k}} + \\
&\quad a_2b_1\vec{\boldsymbol{j}} \times \vec{\boldsymbol{i}} + a_2b_2\vec{\boldsymbol{j}} \times \vec{\boldsymbol{j}} + a_2b_3\vec{\boldsymbol{j}} \times \vec{\boldsymbol{k}} + \\
&\quad a_3b_1\vec{\boldsymbol{k}} \times \vec{\boldsymbol{i}} + a_3b_2\vec{\boldsymbol{k}} \times \vec{\boldsymbol{j}} + a_3b_3\vec{\boldsymbol{k}} \times \vec{\boldsymbol{k}} \\
&= a_1b_1\vec{\boldsymbol{0}} + a_1b_2\vec{\boldsymbol{k}} - a_1b_3\vec{\boldsymbol{j}} \\
&\quad - a_2b_1\vec{\boldsymbol{k}} + a_2b_2\vec{\boldsymbol{0}} + a_2b_3\vec{\boldsymbol{i}} + \\
&\quad a_3b_1\vec{\boldsymbol{j}} - a_3b_2\vec{\boldsymbol{i}} + a_3b_3\vec{\boldsymbol{0}} \\
&= (a_2b_3 - a_3b_2)\vec{\boldsymbol{i}} + (a_3b_1 - a_1b_3)\vec{\boldsymbol{j}} + (a_1b_2 - a_2b_1)\vec{\boldsymbol{k}}
\end{aligned}$$

This is a useful way of remembering how to compute the cross product, particularly when many of the components a_i and b_j are zero.

◀ **44.4 Example.** Compute $\vec{\boldsymbol{k}} \times (p\vec{\boldsymbol{i}} + q\vec{\boldsymbol{j}} + r\vec{\boldsymbol{k}})$:

$$\vec{\boldsymbol{k}} \times (p\vec{\boldsymbol{i}} + q\vec{\boldsymbol{j}} + r\vec{\boldsymbol{k}}) = p(\vec{\boldsymbol{k}} \times \vec{\boldsymbol{i}}) + q(\vec{\boldsymbol{k}} \times \vec{\boldsymbol{j}}) + r(\vec{\boldsymbol{k}} \times \vec{\boldsymbol{k}}) = -q\vec{\boldsymbol{i}} + p\vec{\boldsymbol{j}}.$$

▶

There is another way of remembering how to find $\vec{a} \times \vec{b}$. It involves the "triple product" and determinants. See § 44.3.

44.2. *Algebraic properties of the cross product*

Unlike the dot product, the cross product of two vectors behaves much less like ordinary multiplication. To begin with, the product is ***not commutative*** – instead one has

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \quad \text{for } \textbf{\textit{all}} \text{ vectors } \vec{a} \text{ and } \vec{b}. \tag{66}$$

This property is sometimes called "anti-commutative."

Since the crossproduct of two vectors is again a vector you can compute the cross product of three vectors $\vec{a}, \vec{b}, \vec{c}$. You now have a choice: do you first multiply $\vec{a}$ and $\vec{b}$, or $\vec{b}$ and $\vec{c}$, or $\vec{a}$ and $\vec{c}$? With numbers it makes no difference (e.g. $2 \times (3 \times 5) = 2 \times 15 = 30$ and $(2 \times 3) \times 5 = 6 \times 5 =$ also 30) but with the cross product of vectors it does matter: the cross product is ***not associative,*** i.e.

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c} \quad \text{for } \textbf{\textit{most}} \text{ vectors } \vec{a}, \vec{b}, \vec{c}.$$

$\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$
$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$
so "$\times$" is not associative

The ***distributive law*** does hold, i.e.

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}, \quad \text{and} \quad (\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$$

is true for all vectors $\vec{a}, \vec{b}, \vec{c}$.

Also, an associative law, where one of the factors is a number and the other two are vectors, does hold. I.e.

$$t(\vec{a} \times \vec{b}) = (t\vec{a}) \times \vec{b} = \vec{a} \times (t\vec{b})$$

holds for all vectors $\vec{a}, \vec{b}$ and any number t. We were already using these properties when we multiplied $(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$ in the previous section.

Finally, the cross product is only defined for space vectors, not for plane vectors.

44.3. *The triple product and determinants*

Definition 44.5. *The triple product of three given vectors $\vec{a}, \vec{b}$, and $\vec{c}$ is defined to be*

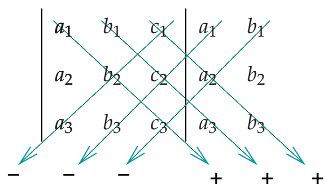
$$\vec{a} \bullet (\vec{b} \times \vec{c}).$$

In terms of the components of $\vec{a}, \vec{b}$, and $\vec{c}$ one has

$$\begin{aligned} \vec{a} \bullet (\vec{b} \times \vec{c}) &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix} \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1. \end{aligned}$$

This quantity is called a ***determinant,*** and is written as follows

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \tag{67}$$



There's a useful shortcut for computing such a determinant: after writing the determinant, append a fourth and a fifth column which are just copies of the first two columns of the determinant. The determinant then is the sum of six products, one for each dotted line in the drawing. Each term has a sign: if the factors are read from top-left to bottom-right, the term is positive, if they are read from top-right to bottom left the term is negative.

This shortcut is also very useful for computing the crossproduct. To compute the cross product of two given vectors $\vec{a}$ and $\vec{b}$ you arrange their components in the following determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & a_1 & b_1 \\ \vec{j} & a_2 & b_2 \\ \vec{k} & a_3 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}. \tag{68}$$

This is not a normal determinant since some of its entries are vectors, but if you ignore that odd circumstance and simply compute the determinant according to the definition (67), you get (68).

An important property of the triple product is that it is much more symmetric in the factors $\vec{a}, \vec{b}, \vec{c}$ than the notation $\vec{a} \bullet (\vec{b} \times \vec{c})$ suggests.

Theorem 44.6. *For any triple of vectors $\vec{a}, \vec{b}, \vec{c}$ one has*

$$\vec{a} \bullet (\vec{b} \times \vec{c}) = \vec{b} \bullet (\vec{c} \times \vec{a}) = \vec{c} \bullet (\vec{a} \times \vec{b}),$$

and

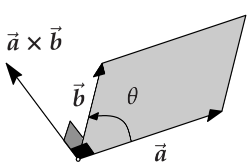
$$\vec{a} \bullet (\vec{b} \times \vec{c}) = -\vec{b} \bullet (\vec{a} \times \vec{c}) = -\vec{c} \bullet (\vec{b} \times \vec{a}).$$

In other words, if you exchange two factors in the product $\vec{a} \bullet (\vec{b} \times \vec{c})$ it changes its sign. If you "rotate the factors," i.e. if you replace $\vec{a}$ by $\vec{b}$, $\vec{b}$ by $\vec{c}$ and $\vec{c}$ by $\vec{a}$, the product doesn't change at all.

44.4. *Geometric description of the cross product*

Theorem 44.7.

$$\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$$



Proof. We use the triple product:

$$\vec{a} \bullet (\vec{a} \times \vec{b}) = \vec{b} \bullet (\vec{a} \times \vec{a}) = \vec{0}$$

since $\vec{a} \times \vec{a} = \vec{0}$ for any vector $\vec{a}$. It follows that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Similarly, $\vec{b} \bullet (\vec{a} \times \vec{b}) = \vec{a} \bullet (\vec{b} \times \vec{b}) = \vec{0}$ shows that $\vec{a} \bullet \vec{b}$ is perpendicular to $\vec{b}$. □

Theorem 44.8.

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \, \sin\theta$$

Proof. Bruce just slipped us a piece of paper with the following formula on it:

$$\|\vec{a} \times \vec{b}\|^2 + (\vec{a} \bullet \vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2.$$

After setting $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and diligently computing both sides we find that this formula actually holds for any pair of vectors $\vec{a}, \vec{b}$! The (long) computation which implies this identity will be presented in class (maybe).

If we assume that Bruce's identity holds then we get

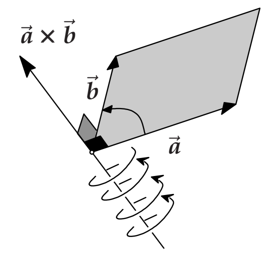
$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \bullet \vec{b})^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2\theta$$

since $1 - \cos^2\theta = \sin^2\theta$. The theorem is proved. □

These two theorems *almost* allow you to construct the cross product of two vectors geometrically. If $\vec{a}$ and $\vec{b}$ are two vectors, then their cross product satisfies the following description:

(1) If $\vec{a}$ and $\vec{b}$ are parallel, then the angle θ between them vanishes, and so their cross product is the zero vector. Assume from here on that $\vec{a}$ and $\vec{b}$ are not parallel.

(2) $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. In other words, since $\vec{a}$ and $\vec{b}$ are not parallel, they determine a plane, and their cross product is a vector perpendicular to this plane.
(3) the length of the cross product $\vec{a} \times \vec{b}$ is $\|\vec{a}\| \cdot \|\vec{b}\| \sin\theta$.



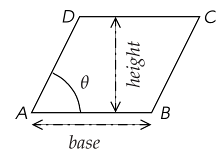
There are only two vectors that satisfy conditions 2 and 3: to determine which one of these is the cross product you must apply the ***Right Hand Rule*** (screwdriver rule, corkscrew rule, etc.) for $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$: if you turn a screw whose axis is perpendicular to $\vec{a}$ and $\vec{b}$ in the direction from $\vec{a}$ to $\vec{b}$, the screw moves in the direction of $\vec{a} \times \vec{b}$.

Alternatively, without seriously injuring yourself, you should be able to make a fist with your ***right*** hand, and then stick out your thumb, index and middle fingers so that your thumb is $\vec{a}$, your index finger is $\vec{b}$ and your middle finger is $\vec{a} \times \vec{b}$. Only people with the most flexible joints can do this with their left hand.

45. **A few applications of the cross product**

45.1. *Area of a parallelogram*

Let $ABCD$ be a parallelogram. Its area is given by "height times base," a formula which should be familiar from high school geometry.



If the angle between the sides AB and AD is θ, then the height of the parallelogram is $\|\overrightarrow{AD}\| \sin\theta$, so that the area of $ABCD$ is

$$\text{area of } ABCD = \|\overrightarrow{AB}\| \cdot \|\overrightarrow{AD}\| \sin\theta = \|\overrightarrow{AB} \times \overrightarrow{AD}\|. \tag{69}$$

The area of the triangle ABD is of course half as much,

$$\text{area of triangle } ABD = \tfrac{1}{2} \|\overrightarrow{AB} \times \overrightarrow{AD}\|.$$

These formulae are valid even when the points A, B, C, and D are points in space. Of course they must lie in one plane for otherwise $ABCD$ couldn't be a parallelogram.

◀ **45.1 Example.** Let the points $A(1,0,2)$, $B(2,0,0)$, $C(3,1,-1)$ and $D(2,1,1)$ be given.

Show that ABCD is a parallelogram, and compute its area.

Solution: $ABCD$ will be a parallelogram if and only if $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$. In terms of the position vectors $\vec{a}$,$\vec{b}$, $\vec{c}$ and $\vec{d}$ of A, B, C, D this boils down to

$$\vec{c} - \vec{a} = (\vec{b} - \vec{a}) + (\vec{d} - \vec{a}), \quad \text{i.e.} \quad \vec{a} + \vec{c} = \vec{b} + \vec{d}.$$

For our points we get

$$\vec{a} + \vec{c} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix}, \qquad vb + \vec{d} = \begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}4\\1\\1\end{pmatrix}.$$

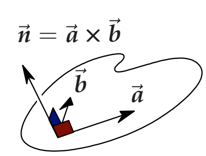
So $ABCD$ is indeed a parallelogram. Its area is the length of

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{pmatrix}2-1\\0\\0-2\end{pmatrix} \times \begin{pmatrix}2-2\\1-0\\1-0\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix} \times \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}-2\\-1\\-1\end{pmatrix}.$$

So the area of $ABCD$ is $\sqrt{(-2)^2 + (-1)^2 + (-1)^2} = \sqrt{6}$. ▶

45.2. *Finding the normal to a plane*

If you know two vectors $\vec{a}$ and $\vec{b}$ which are parallel to a given plane $\mathcal{P}$ but not parallel to each other, then you can find a normal vector for the plane $\mathcal{P}$ by computing



$$\vec{n} = \vec{a} \times \vec{b}.$$

We have just seen that the vector $\vec{n}$ must be perpendicular to both $\vec{a}$ and $\vec{b}$, and hence[9] it is perpendicular to the plane $\mathcal{P}$.

[9] This statement needs a proof which we will skip. Instead have a look at the picture

This trick is especially useful when you have three points A, B and C, and you want to find the defining equation for the plane $\mathcal{P}$ through these points. We will assume that the three points do not all lie on one line, for otherwise there are many planes through A, B and C.

To find the defining equation we need one point on the plane (we have three of them), and a normal vector to the plane. A normal vector can be obtained by computing the cross product of two vectors parallel to the plane. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are both parallel to $\mathcal{P}$, the vector $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ is such a normal vector.

Thus the defining equation for the plane through three given points A, B and C is

$$\vec{n} \bullet (\vec{x} - \vec{a}) = 0, \quad \text{with} \quad \vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}).$$

◄ **45.2 Find the defining equation of the plane $\mathcal{P}$ through the points $A(2, -1, 0)$, $B(2, 1, -1)$ and $C(-1, 1, 1)$. Find the intersections of $\mathcal{P}$ with the three coordinate axes, and find the distance from the origin to $\mathcal{P}$.**

Solution: We have

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

so that

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$$

is a normal to the plane. The defining equation for $\mathcal{P}$ is therefore

$$0 = \vec{n} \bullet (\vec{x} - \vec{a}) = \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix} \bullet \begin{pmatrix} x_1 - 2 \\ x_2 + 1 \\ x_3 - 0 \end{pmatrix}$$

i.e.

$$4x_1 + 3x_2 + 6x_3 - 5 = 0.$$

The plane intersects the x_1 axis when $x_2 = x_3 = 0$ and hence $4x_1 - 5 = 0$, i.e. in the point $(\frac{5}{4}, 0, 0)$. The intersections with the other two axes are $(0, \frac{5}{3}, 0)$ and $(0, 0, \frac{5}{6})$.

The distance from any point with position vector $\vec{x}$ to $\mathcal{P}$ is given by

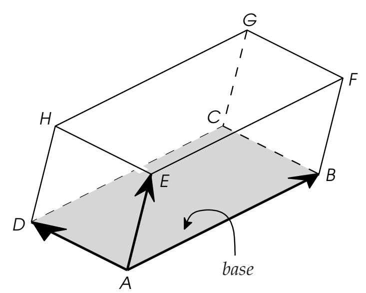
$$\text{dist} = \pm \frac{\vec{n} \bullet (\vec{x} - \vec{a})}{\|\vec{n}\|},$$

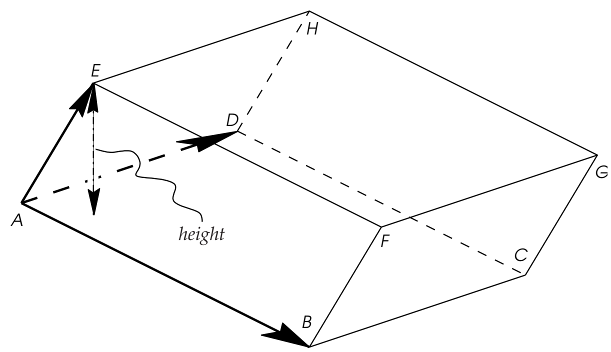
so the distance from the origin (whose position vector is $\vec{x} = \vec{0} = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$) to $\mathcal{P}$ is

$$\text{distance origin to } \mathcal{P} = \pm \frac{\vec{a} \bullet \vec{n}}{\|\vec{n}\|} = \pm \frac{2 \cdot 4 + (-1) \cdot 3 + 0 \cdot 6}{\sqrt{4^2 + 3^2 + 6^2}} = \frac{5}{\sqrt{61}} (\approx 1.024 \cdots).$$

►

45.3. *Volume of a parallelepiped*





A ***parallelepiped*** is a three dimensional body whose sides are parallelograms. For instance, a cube is an example of a parallelepiped; a rectangular block (whose faces are rectangles, meeting at right angles) is also a parallelepiped. Any parallelepiped has 8 vertices (corner points), 12 edges and 6 faces.

Let $\genfrac{}{}{0pt}{}{ABCD}{EFGH}$ be a parallelepiped. If we call one of the faces, say $ABCD$, the base of the parallelepiped, then the other face $EFGH$ is parallel to the base. The ***height of the parallelepiped*** is the distance from any point in $EFGH$ to the base, e.g. to compute the height of $\genfrac{}{}{0pt}{}{ABCD}{EFGH}$ one could compute the distance from the point E (or F, or G, or H) to the plane through $ABCD$.

The volume of the parallelepiped $\genfrac{}{}{0pt}{}{ABCD}{EFGH}$ is given by the formula

$$\text{Volume}\ \begin{matrix}ABCD\\EFGH\end{matrix} = \text{Area of base} \times \text{height}.$$

Since the base is a parallelogram we know its area is given by

$$\text{Area of base}ABCD = \|\overrightarrow{AB} \times \overrightarrow{AD}\|$$

We also know that $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AD}$ is a vector perpendicular to the plane through $ABCD$, i.e. perpendicular to the base of the parallelepiped. If we let the angle between the edge AE and the normal $\vec{n}$ be ψ, then the height of the parallelepiped is given by

$$\text{height} = \|\overrightarrow{AE}\| \cos\psi.$$

Therefore the triple product of $\overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AE}$ is

$$\begin{aligned}\text{Volume}\ \begin{matrix}ABCD\\EFGH\end{matrix} &= \text{height} \times \text{Area of base}\\ &= \|\overrightarrow{AE}\| \cos\psi \quad \|\overrightarrow{AB} \times \overrightarrow{AD}\|\,,\end{aligned}$$

i.e.

$$\boxed{\text{Volume}\ \begin{matrix}ABCD\\EFGH\end{matrix} = \overrightarrow{AE} \bullet (\overrightarrow{AB} \times \overrightarrow{AD}).}$$

46. **Notation**

In the next chapter we will be using vectors, so let's take a minute to summarize the concepts and notation we have been using.

Given a point in the plane, or in space you can form its position vector. So associated to a point we have three different objects: the point, its position vector and its coordinates. here is the notation we use for these:

OBJECT	NOTATION
Point	Upper case letters, A, B, etc.
Position vector	Lowercase letters with an arrow on top. The position vector $\overrightarrow{OA}$ of the point A should be $\vec{a}$, so that letters match across changes from upper to lower case.
Coordinates of a point	The coordinates of the point A are the same as the components of its position vector $\vec{a}$: we use lower case letters with a subscript to indicate which coordinate we have in mind: (a_1, a_2).

47. PROBLEMS

Computing and drawing vectors

361. Simplify the following

$$\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix};$$

$$\vec{b} = 12 \begin{pmatrix} 1 \\ 1/3 \end{pmatrix} - 3 \begin{pmatrix} 4 \\ 1 \end{pmatrix};$$

$$\vec{c} = (1+t) \begin{pmatrix} 1 \\ 1-t \end{pmatrix} - t \begin{pmatrix} 1 \\ -t \end{pmatrix},$$

$$\vec{d} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

362. If $\vec{a}, \vec{b}, \vec{c}$ are as in the previous problem, then which of the following expressions mean anything? Compute those expressions that are well defined.

(*i*) $\vec{a} + \vec{b}$ (*ii*) $\vec{b} + \vec{c}$ (*iii*) $\pi\vec{a}$

(*iv*) $\vec{b}^2$ (*v*) $\vec{b}/\vec{c}$ (*vi*) $\|\vec{a}\| + \|\vec{b}\|$

(*vii*) $\|\vec{b}\|^2$ (*viii*) $\vec{b}/\|\vec{c}\|$

363. Let $\vec{u}, \vec{v}, \vec{w}$ be three given vectors, and suppose

$$\vec{a} = \vec{v} + \vec{w}, \quad \vec{b} = 2\vec{u} - \vec{w}, \quad \vec{c} = \vec{u} + \vec{v} + \vec{w}.$$

(a) Simplify $\vec{p} = \vec{a} + 3\vec{b} - \vec{c}$ and $\vec{q} = \vec{c} - 2(\vec{u} + \vec{a})$.

(b) Find numbers r, s, t such that $r\vec{a} + s\vec{b} + t\vec{c} = \vec{u}$.

(c) Find numbers k, l, m such that $k\vec{a} + l\vec{b} + m\vec{c} = \vec{v}$.

364. Prove the Algebraic Properties (49), (50), (51), and (52) in section 40.2.

365. (a) Does there exist a number x such that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}?$$

(b) Make a drawing of all points P whose position vectors are given by

$$\vec{p} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} x \\ x \end{pmatrix}.$$

(c) Do there exist a numbers x and y such that

$$x \begin{pmatrix} 1 \\ 2 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}?$$

366. Given points $A(2,1)$ and $B(-1,4)$ compute the vector $\overrightarrow{AB}$. *Is* $\overrightarrow{AB}$ *a position vector?*

367. Given: points $A(2,1)$, $B(3,2)$, $C(4,4)$ and $D(5,2)$. *Is $ABCD$ a parallelogram?*

368. Given: points $A(0,2,1)$, $B(0,3,2)$, $C(4,1,4)$ and D.

(a) If $ABCD$ is a parallelogram, then what are the coordinates of the point D?

(b) If $ABDC$ is a parallelogram, then what are the coordinates of the point D?

369. You are given three points in the plane: A has coordinates $(2,3)$, B has coordinates $(-1,2)$ and C has coordinates $(4,-1)$.

(a) Compute the vectors $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{BC}$ and $\overrightarrow{CB}$.

(b) Find the points P, Q, R and S whose position vectors are $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, and $\overrightarrow{BC}$, respectively. *Make a precise drawing in figure 21.*

370. Have a look at figure 22

(a) Draw the vectors $2\vec{v} + \frac{1}{2}\vec{w}$, $-\frac{1}{2}\vec{v} + \vec{w}$, and $\frac{3}{2}\vec{v} - \frac{1}{2}\vec{w}$

(b) Find real numbers s, t such that $s\vec{v} + t\vec{w} = \vec{a}$.

(c) Find real numbers p, q such that $p\vec{v} + q\vec{w} = \vec{b}$.

(d) Find real numbers k, l, m, n such that $\vec{v} = k\vec{a} + l\vec{b}$, and $\vec{w} = m\vec{a} + n\vec{w}$.