INHOMOGENEOUS LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

1. Definitions and a general fact

If $A$ is an $n \times n$ matrix and $f(t)$ is some given vector function, then the system of differential equations

$$x'(t) - Ax(t) = f(t)$$

is said to be linear inhomogeneous. The given Right Hand Side $f(t)$ is sometimes called the “forcing term.” The corresponding homogeneous equation is the same equation in which $f(t)$ has been replaced by zero:

$$x'(t) - Ax(t) = 0.$$  \hspace{1cm} (2)

1.1. Theorem. The general solution of the inhomogeneous system of equations (1) is

$$x(t) = x_h(t) + x_p(t),$$

where $x_h(t)$ is the general solution to the homogeneous equation, and $x_p(t)$ is any particular solution to the inhomogeneous equation.

Since we know how to solve the homogeneous equation using the eigenvalues and eigenvectors of $A$, “all we still have to do” is find a particular solution.

2. Finding Particular Solutions by Guessing

2.1. Exponential forcing terms. If the “forcing term” $f(t)$ is of the form

$$f(t) = e^{\alpha t}v$$

where $\alpha$ is a constant, and $v$ is a constant vector, then try to find a solution of the form

$$x_p(t) = e^{\alpha t}w$$

Substitution in the diffeq leads to

$$\alpha e^{\alpha t}w - A(e^{\alpha t}w) = e^{\alpha t}v.$$  \hspace{1cm} (3)

Cancelling exponentials then gives you the following equation for the vector $w$

$$(\alpha I - A)w = v.$$  \hspace{1cm} (4)

If $\alpha$ is not an eigenvalue of $A$ then this system can be solved, and the solution is

$$w = (\alpha I - A)^{-1}v$$

so that the particular solution of the diffeq is

$$x_p(t) = e^{\alpha t}(\alpha I - A)^{-1}v$$

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2.2. Trigonometric forcing terms. If the forcing term is
\[ f(t) = \cos(\omega t) a + \sin(\omega t) b \]
then most of the time a particular solution of the same form can be found.

You can substitute a particular solution of the form
\[ x_p(t) = \cos(\omega t) p + \sin(\omega t) q \]
and solve for the vectors \( p \) and \( q \). The resulting algebra becomes complicated and
it is better to use a detour through complex numbers.

The trick to remember is that the forcing term
\[ f(t) = \cos(\omega t) a + \sin(\omega t) b \]
is the real part of
\[ \cos(\omega t) a + \sin(\omega t) b + i \{ \sin(\omega t) a + \cos(\omega t) b \}, \]
which can be written more shortly as
\[ (a - ib)(\cos \omega t + i \sin \omega t) = ce^{i\omega t} \]
if \( c = a - ib \).

So instead of solving the original equation \( x_p' - Ax_p = f(t) \) you solve the complex
version of this equation, namely
\[ z_p' - Az_p = ce^{i\omega t}. \]
Then the real part
\[ x_p(t) = \Re[z_p(t)] \]
is a solution of the original equation \( x' - Ax = f(t) \).

You find \( z_p(t) \) as in the previous case, namely, try
\[ z_p(t) = e^{i\omega t} d \]
where \( d \) is an undetermined complex vector. Substitution yields
\[ (i\omega I - A) d = c \implies d = (i\omega I - A)^{-1} c. \]
Thus \( d \) can be computed by solving the system of equations \( (i\omega I - A) d = c \), e.g.
by row reducing. The solution will be a some complex vector \( d \). Multiply with \( e^{i\omega t} \), using Euler’s formula, and take the real part to get the real particular solution
\[ x_p(t) = \Re[e^{i\omega t}d]. \]

3. Variation of Constants
3.1. Description of the general method. Suppose you have already solved the
homogeneous equation \( x' - Ax = 0 \), and that your general solution is
\[ x(t) = c_1 x_1(t) + \cdots + c_n x_n(t). \]
If you now want to find the solution to the inhomogeneous equation
\[ x'(t) - Ax(t) = f(t) \]
then there is a trick, called variation of constants or variation of parameters
that will give you that solution.

The trick is to look for a solution of the form
\[ x(t) = c_1(t)x_1(t) + \cdots + c_n(t)x_n(t). \]
This is almost the same as the general solution to the homogeneous equation, except here the quantities $c_1, \ldots, c_n$ are no longer constants: we allow them to vary, and this gives the method its name.

The method consists of substituting (7) in the inhomogeneous equation, and deriving equations for the “varying constants” $c_1(t), \ldots, c_n(t)$. When you do the substitution, many terms will cancel, and you always end up with the following:

$$x'_p(t) - Ax_p(t) = c'_1(t)x_1(t) + \cdots + c'_n(t)x_n(t).$$

Since we want the right hand side $x'_p - Ax_p$ to equal the forcing term $f(t)$, so we must solve the equation(s)

$$c'_1(t)x_1(t) + \cdots + c'_n(t)x_n(t) = f(t)$$

for the derivatives $c'_1(t), \ldots, c'_n(t)$.

Once we have found $c'_1(t), \ldots, c'_n(t)$ we can integrate to find the “varying constants” $c_1(t), \ldots, c_n(t)$, and therefore also the solution $x_p(t)$.

### 3.2. When you have $n$ eigenvectors and eigenvalues

Typically you will have solved the homogeneous equation by computing the eigenvalues

$$\lambda_1, \ldots, \lambda_n$$

and corresponding eigenvectors

$$v_1, \ldots, v_n$$

of the matrix $A$ (So $Av_1 = \lambda_1 v_1$, etc.). If this is the case then $x_j(t) = e^{\lambda_j t}v_j$ is a solution, and the general solution you found is

$$x(t) = c_1 e^{\lambda_1 t}v_1 + \cdots + c_n e^{\lambda_n t}v_n.$$ 

In this case the method of Variation of Constants tells you to try a particular solution of the form

$$x(t) = c_1(t) e^{\lambda_1 t}v_1 + \cdots + c_n(t) e^{\lambda_n t}v_n,$$

where the constants $c_i$ have been replaced by functions of $t$.

Substituting $x(t)$ in the equation $x' - Ax = f$ is made easy by using the handy formula (8) which, in the present case, says

$$x' - Ax = c'_1(t)e^{\lambda_1 t}v_1 + \cdots + c'_n(t)e^{\lambda_n t}v_n.$$ 

We now must set this equal to the forcing term $f(t)$. To do this we write the forcing term as a linear combination of the eigenvectors

$$f(t) = f_1(t)v_1 + \cdots + f_n(t)v_n.$$ 

Finding $f_1(t), \ldots, f_n(t)$ requires you to solve a system of $n$ linear equations, which you can do by row reduction.

The equation $x' - Ax = f$ together with (10) and (11) then implies

$$(12) c'_1(t)e^{\lambda_1 t} = f_1(t), \quad \ldots \quad c'_n(t)e^{\lambda_n t} = f_n(t),$$
and thus
\[ c_1(t) = \int e^{-\lambda_1 t} f_1(t) \, dt, \quad \ldots \quad c_n(t) = \int e^{-\lambda_n t} f_n(t) \, dt. \]
Reinsert these formulas for \( c_i(t) \) in the original definition of \( x(t) \) and you get
\[ x(t) = \left[ \int e^{-\lambda_1 t} f_1(t) \, dt \right] e^{\lambda_1 t} v_1 + \cdots + \left[ \int e^{-\lambda_n t} f_n(t) \, dt \right] e^{\lambda_n t} v_n \] (13)

The integrals between brackets \([\cdots]\) are indefinite integrals, and therefore contain constants. So the solution \( x(t) \) in this formula has \( n \) constants in it, and it is actually the general solution to the inhomogeneous equation. If all you want is a formula for the general solution then you could stop here: (13) is your solution.

3.3. The particular solution with zero initial data. There is one particular solution which is of interest, namely, the solution \( x(t) \) which starts out with \( x(0) = 0 \). Why? Suppose that you had to solve the initial value problem
\[
\begin{cases}
x'(t) - Ax(t) = f(t) \\
x(0) = a
\end{cases}
\]
where \( a \) is the given initial value. As before, we write the answer as
\[ x(t) = x_h(t) + x_p(t), \] (14)
but now we also specify the initial values of both \( x_h(t) \) and \( x_p(t) \): let \( x_h \) be the solution of
\[ x_h'(t) - A x_h(t) = 0, \quad x_h(0) = a \] (15)
and let \( x_p(t) \) be the solution of
\[ x_p'(t) - A x_p(t) = f(t), \quad x_p(0) = 0 \] (16)
If you think of the solution \( x(t) \) as the “response” to the “input” \( f(t) \), then equation (14) splits the response \( x(t) \) into a component \( x_h(t) \) which is due to the initial conditions, and another component \( x_p(t) \) which is due to the forcing term.

You can find \( x_h(t) \) by solving the homogeneous equation and choosing the constants \( c_i \) so that \( x_h(0) = a \).

You can find the forcing term by using Variation of Constants as above, and requiring \( c_1(0) = 0, \ldots, c_n(0) = 0 \). Then the equations (12) (and the Fundamental Theorem of Calculus)\(^1\) imply
\[
\begin{cases}
  c_1(t) = \int_0^t c_1'(s) \, ds = \int_0^t e^{-\lambda_1 s} f_1(s) \, ds \\
  \vdots \\
  c_n(t) = \int_0^t e^{-\lambda_n s} f_n(s) \, ds
\end{cases}
\] (17)
Finally you substitute these integrals in the definition (21) of \( x_p(t) \), and you find
\[ x_p(t) = \left[ \int_0^t e^{-\lambda_1 s} f_1(s) \, ds \right] e^{\lambda_1 t} v_1 + \cdots + \left[ \int_0^t e^{-\lambda_n s} f_n(s) \, ds \right] e^{\lambda_n t} v_n \] (18)
\[ c_i(t) \]
\[ c_n(t) \]
\[ \int_0^t c'(s) \, ds \]
\[ \int_0^t e^{-\lambda s} f(s) \, ds \]
\[
\text{1. The Fundamental Theorem of Calculus says } c(t) - c(0) = \int_0^t c'(s) \, ds, \text{ so if } c(0) = 0 \text{ then } c(t) = \int_0^t c'(s) \, ds
\]
One often rewrites the coefficients of the vectors $v_i$ as follows:

\[
\int_0^t e^{-\lambda_1 s} f_1(s) \, ds \mid_{c_i(t)} = \int_0^t e^{\lambda_1 t} e^{-\lambda_1 s} f_1(s) \, ds = \int_0^t e^{\lambda_1 (t-s)} f_1(s) \, ds.
\]

The particular solution with zero initial data is then

\[
x_p(t) = \left[ \int_0^t e^{\lambda_1 (t-s)} f_1(s) \, ds \right] v_1 + \cdots + \left[ \int_0^t e^{\lambda_n (t-s)} f_n(s) \, ds \right] v_n.
\]

### 3.4. Convolution integrals.

Integrals of the form

\[
\int_0^t g(t-s) f(s) \, ds
\]

are called **convolution integrals**, and they show up very often when you solve constant coefficient linear differential equations.

The coefficient

\[
\int_0^t e^{\lambda_1 (t-s)} f_1(s) \, ds.
\]

of $v_1$ in the particular solution $x_p$ is a convolution integral with $g(t) = e^{\lambda_1 t}$. It has the following interpretation: the $v_1$ component of the response is given by Since $f_1(s)$ is the $v_1$ component of the input at time $s$, the $v_1$ component of the solution is an average of the $v_1$ components $f_1(s)$ of the input at earlier times $s$. The weight with which $f_1(s)$ contributes to the solution at time $t$ only depends on the time delay $t-s$.

### 4. An example

Let $y(t)$ be the solution of

\[
\begin{align*}
y''(t) + 3y'(t) + 2y(t) &= F(t), \\
y(0) = y'(0) &= 0.
\end{align*}
\]

First we rewrite this as a system of first order equations: let $x_1 = y$, $x_2 = y'$.

Then

\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= -2x_1 - 3x_2 - F(t),
\end{align*}
\]

so $x = (x_1, x_2)$ satisfies

\[
x'(t) = Ax(t) + f(t) \text{ with } A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \text{ and } f(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}.
\]

The eigenvalues and vectors of $A$ are

\[
\lambda_1 = -1, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \lambda_2 = -2, \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]

Therefore we should find a particular solution of the form

\[
x_p(t) = c_1(t)e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2(t)e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]

The initial conditions are

\[
x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0.
\]
We get these initial conditions if we require
\begin{equation}
(22) \quad c_1(0) = c_2(0) = 0
\end{equation}
Substituting \( x_p(t) \) from (21) in \( x' - Ax \) leads to
\begin{equation}
(23) \quad x'_p(t) - Ax_p(t) = c'_1(t)e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c'_2(t)e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\end{equation}
by the handy identity (10). To compare this with \( f(t) \) we must write the forcing term \( f(t) \) as a combination of the eigenvectors \( v_1, v_2: \)
\[ f(t) = \begin{pmatrix} 0 \\ F(t) \end{pmatrix} = f_1(t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + f_2(t) \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 \\ -f_1 - f_2 \end{pmatrix}. \]
Solve this by row reduction:
\[
\begin{bmatrix}
1 & 1 & 0 \\
-1 & -2 & F(t)
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & F(t) \\
0 & -F(t)
\end{bmatrix} \quad \Rightarrow \quad f_1(t) = F(t), f_2(t) = -F(t).
\]
Our decomposition of the forcing term \( f(t) \) into the eigenvectors \( v_1, v_2 \) is therefore
\begin{equation}
(24) \quad f(t) = F(t)v_1 - F(t)v_2
\end{equation}
Now combining (24) and (23) we see that \( x' - Ax = f \) implies
\[ c'_1(t)e^{-t}v_1 + c'_2(t)e^{-2t}v_2 = F(t)v_1 - F(t)v_2 \]
so that
\[ c'_1(t)e^{-t} = F(t) \text{ and } c'_2(t)e^{-2t} = -F(t). \]
Solving for \( c'_1 \) and \( c'_2 \), and integrating, taking into account that \( c_1(0) = c_2(0) = 0 \), you get
\[ c_1(t) = \int_0^t e^sF(s) \, ds \text{ and } c_2(t) = -\int_0^t e^{2s}F(s) \, ds. \]
Put this back into the solution \( x_p(t): \)
\[ x_p(t) = c_1(t)e^{-t}v_1 + c_2(t)e^{-2t}v_2 = \left[ \int_0^t e^{s-t}F(s) \, ds \right]v_1 - \left[ \int_0^t e^{2(s-t)}F(s) \, ds \right]v_2 \]
To see the solution in all its components remember that \( v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \), so that
\[ x_p(t) = \left[ \int_0^t e^{s-t}F(s) \, ds \right] \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \left[ \int_0^t e^{2(s-t)}F(s) \, ds \right] \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \]
Adding these vectors produces a large formula. Since \( x(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} \) the top component of \( x_p(t) \) is the solution \( y(t) \) to the second order equation we began with. It is given by
\[ y(t) = \int_0^t e^{s-t}F(s) \, ds - \int_0^t e^{2(s-t)}F(s) \, ds = \int_0^t \left\{ e^{-(t-s)} - e^{-2(t-s)} \right\} F(s) \, ds. \]
This integral is of the form
\[ \int_0^t K(t - s)F(s) \, ds \]
where
\[ K(\tau) = e^{-\tau} - e^{-2\tau}. \]
The interpretation of the convolution integral is that the solution $y(t)$ is a response to all the inputs $F(s)$ for $0 < s < t$, and that the contribution of the input $F(s)$ at time $s$ is weighed by a factor $K(t - s)$.

A graph of $K(\tau)$; $K(\tau)$ is maximal at $\tau = \ln 2$

5. PROBLEMS

1. Use the method of variation of Constants to find the solution to $y'' + 5y' + 6y = F(t)$, with $y(0) = y'(0) = 0$. More specifically:
   
   (i) rewrite the equation as a system of first order differential equations of the form $x' = Ax + f(t)$; what are $A$ and $f(t)$ in this problem?
   
   (ii) Find the eigenvalues and eigenvectors of $A$.
   
   (iii) Find the general solution to the homogeneous equation $x' = Ax$.
   
   (iv) Which kind of solution do you try when you use V.o.C. to find a particular solution?
   
   (v) which initial conditions should you impose on $x$ if you want $y(0) = y'(0) = 0$?
   
   (vi) Find the solution $x(t)$.
   
   (vii) Find the solution $y(t)$.

2. Do the same for $y'' - y = F(t)$, with $y(0) = y'(0) = 0$.

3. Use V.o.C. to find the solution to

$$\frac{dx}{dt} = Ax + f(t), \quad \text{with} \quad A = \begin{pmatrix} -10 & 0 & 0 \\ 10 & -5 & 0 \\ 0 & 5 & -4 \end{pmatrix}, \quad \text{and} \quad f(t) = \begin{pmatrix} F(t) \\ 0 \\ 0 \end{pmatrix}. $$

Write a formula for $x_3(t)$ (the third component of $x(t)$): the answer will be a convolution integral of the form

$$\int_0^t K(t - s)F(s) \, ds$$

Identify $K(t - s)$.

(There is an interpretation of this problem as in example 2 on page 496: in the present problem the inflow of brine into the first tank is given by $F(t)$, rather than just “20” as in the book.)

6. PROOF OF FORMULA (8)

The chain rule tells us that

$$x'(t) = c_1(t)x_1'(t) + \cdots + c_n(t)x_n'(t)$$

$$+ c'_1(t)x_1(t) + \cdots + c'_n(t)x_n(t).$$
Since the $x_j(t)$ are solutions to the homogeneous equation, we get
\[ x'(t) = c_1(t)Ax_1(t) + \cdots + c_n(t)Ax_n(t) \]
\[ + c'_1(t)x_1(t) + \cdots + c'_n(t)x_n(t). \]
On the other hand we also have
\[ Ax(t) = c_1(t)Ax_1(t) + \cdots + c_n(t)Ax_n(t), \]
and therefore
\[ x'(t) - Ax(t) = c'_1(t)x_1(t) + \cdots + c'_n(t)x_n(t). \]