1 Find the general solution of the system of differential equations \( x' = Ax \), for each of the following matrices

\[
A_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\
A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & -4 & 0 \\ 4 & 0 & -3 \\ 0 & 3 & 0 \end{pmatrix} \\
A_5 = \begin{pmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A_6 = \begin{pmatrix} -1 & 2 & 0 \\ -2 & -1 & 0 \\ a & b & 1 \end{pmatrix}
\]

in which \( a \) and \( b \) are constants.

2 If \( B = e^{2A_1}e^{-3A_2} \), then what is \( \det B \)?

3 Find at least one solution to the following differential equations

\[
x'(t) = A_1 x(t) + \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \\
x'(t) = A_4 x(t) + \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}
\]

\[
x'(t) = A_3 x(t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
x'(t) = A_4 x(t) + \begin{pmatrix} 4 \\ 0 \end{pmatrix}
\]

\[
x'(t) = A_4 x(t) + e^{-t} e_2 \\
x'(t) = A_4 x(t) + t e_2
\]

Here the matrices \( A_i \) are the same as in problem 1.

4 (i) For which of the matrices \( A_i \) in problem 1 is \( e^{tA_1} \) a symmetric matrix for all \( t \in \mathbb{R} \)?

(ii) For which of the matrices \( A_i \) in problem 1 is \( e^{tA_i} \) an orthogonal matrix for all \( t \in \mathbb{R} \)?

5 Let \( A = A_4 \) from problem 1, and let \( x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). Show that \( \|x\| = \|e^{tA}x\| \) for all \( t \in \mathbb{R} \).

6 Suppose that for some \( n \times n \) matrix \( A \) there is a vector \( x \in \mathbb{R}^n \) such that \( e^{tA}x = x \) holds for all \( t \in \mathbb{R} \). Show that \( x \in \ker A \).

That’s all . . .