1

Consider the surface parametrization $\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\vec{r}(u, v) = \begin{pmatrix} u \\ uv \\ v^2 \end{pmatrix}.$$  

(i) At which points is the parametrization $\vec{r}$ smooth ("regular")?

(ii) At which point(s) is the tangent plane to the surface parallel to the plane $4x + 2y + z = 0$?

Consider the surface $S = \{ \vec{r}(u, v) \mid 1 \leq u, v, \leq 2 \}$, with $\vec{r}$ as above.

(iii) Express the area of $S$ as a double integral of the type $\iint_R f(u, v) \, du \, dv$, where you explicitly state what $R$ and $f(u, v)$ are.

2

Let $\mathcal{R}$ be the rectangle $[1, 2] \times [1, 2]$, and let $\mathcal{D}$ be the image of $\mathcal{R}$ under the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(u, v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \end{pmatrix} = \begin{pmatrix} u/v \\ uv \end{pmatrix}.$$  

(i) Sketch $\mathcal{D}$.

(ii) Compute the area of $\mathcal{D}$.

[ Problems 3 & 4 on the back of this sheet ]

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SCORE • 1: 2: 3: 4: Total:
Let $\mathcal{R} \subset \mathbb{R}^3$ be the three dimensional solid given by
$$\mathcal{R} = \{(x, y, z) \mid x^2 + y^2 \leq R^2, 0 \leq z \leq x^2\}$$
in which $R > 0$ is a constant.

(i) Sketch $\mathcal{R}$.

(ii) The moment of inertia about the $z$ axis of $\mathcal{R}$ is defined by
$$I_z = \iiint_{\mathcal{R}} (x^2 + y^2) \, dx\, dy\, dz$$
Compute $I_z$ by choosing convenient coordinates.

4

(i) Consider the vector fields
$$\vec{v}(x, y, z) = -y\vec{i} + x\vec{j} + z\vec{k}.$$  

Use Stokes' Theorem to compute
$$\int \int_{\alpha} \vec{v} \cdot \vec{t} \, ds,$$
where $\alpha$ is the triangle whose corners are the origin, $(1, 0, 0)$ and $(1, 2, 0)$. (Choose your own orientation for $\alpha$.)

(ii) Let $\mathcal{R} \subset \mathbb{R}^3$ be any bounded region with smooth boundary $S$. Let $\vec{n}$ denote the outward unit normal to $S$. Consider the vector field
$$\vec{w}(x, y, z) = x\vec{i} + y^3\vec{j} + z^5\vec{k}.$$  

Using Green’s Theorem, decide if the following statement is **True or False**: The surface integral
$$\Phi \overset{\text{def}}{=} \int \int_S \vec{w} \cdot \vec{n} \, dS$$
can never be negative.