MATH 376
Concerning the 2nd midterm

The subjects of this midterm are:

- Line integrals (again). See the review sheet from the 1st midterm.
- The transformation formula for double integrals
- Surface integrals
- Stokes’ Theorem

When studying for this exam you should do the problems from the homework, and those listed below. You should also study a few proofs (to be listed below).

Some sample problems

1  [Transformation formula] Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map given by \( F(u, v) = (2uv, u^2 - v^2) \).

(i) Let \( D \) be the image of the rectangle \( R = \{(u, v) \mid a \leq u \leq b, a \leq v \leq b\} \). Draw \( D \).

(ii) Compute the area of \( D \).

(iii) Compute the volume of the solid obtained by rotating the region \( D \) around the \( y \)-axis.

(iv) Find the moments of inertia of \( D \). (Look the formulas up in the book!)

2  [Integral in polar, spherical & cylindrical coordinates] Let \( S \) be the spherical shell \( S_{a,b} = \{(x, y, z) \mid a^2 \leq x^2 + y^2 + z^2 \leq b^2\} \).

(i) Compute the volume and moment of inertia

\[
I = \iiint_S (x^2 + y^2) \, dxdydz.
\]

(ii) Compute the center of mass of the northern half of the shell, i.e. of \( S^+_{a,b} = \{(x, y, z) \in S_{a,b} \mid z \geq 0\} \).

3  [Parametrization of a surface and tangent planes] Consider the parametrization

\[
\vec{r}(u, v) = uv\vec{i} + v^2\vec{j} + (u^2 - v^2)\vec{k}
\]

(i) At which points in the \((u, v)\)-plane is the parametrization \( \vec{r} \) regular (i.e. “smooth” in Apostol’s terminology – look up the definition)?

(ii) Find the tangent plane at any smooth point \( \vec{r}(\bar{u}, \bar{v}) \) on the surface.

(iii) At which point(s) \( \vec{r}(\bar{u}, \bar{v}) \) is the tangent plane parallel to the plane with equation \( 2x + y + z = \frac{355}{113} \)?

(iv) At which point(s) \( \vec{r}(\bar{u}, \bar{v}) \) is the tangent plane parallel to the line through \((1, 2, 0)\) and \((0, 0, 1)\)?

(v) At which point(s) \( \vec{r}(\bar{u}, \bar{v}) \) does the tangent plane contain the origin?

4  [Surface integrals] Find the center of mass of the part of the sphere with radius \( a > 0 \) contained in the 1st octant, i.e. of \( S = \{(x, y, z) \mid x, y, z \geq 0, x^2 + y^2 + z^2 = a^2\} \).
Let \( S \) be the triangle with corners \( A = (a, 0, 0) \), \( B = (0, b, 0) \) and \( C = (0, 0, c) \) (assume \( a, b, c > 0 \)). We orient \( S \) so that its normal vector points upwards (has positive \( z \) component).

Let \( T \) be the boundary of \( S \).

(i) Find parametrizations for \( S \) and \( T \).

(ii) Compute the area and center of mass of the triangle (see book for definitions.)

(iii) Compute \( \iint_S \vec{v} \cdot \vec{n} \, dS \) where \( \vec{v}(x, y, z) = i - 2j \).

(iv) Compute \( \int_T \vec{v} \cdot \vec{t} \, ds \) where \( \vec{v}(x, y, z) = y\vec{i} - x\vec{j} + z\vec{k} \) in two different ways: first directly from the definitions, and again using Stokes' Theorem.

Let the surface \( S \) be the graph of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) which is given in polar coordinates by \( z = F(r, \theta) \), \( a \leq r \leq b \), \( \alpha \leq \theta \leq \beta \) (i.e. \( f(x, y) = F(r(x, y), \theta(x, y)) \) where \( r(x, y) = \sqrt{x^2 + y^2} \), \( \tan \theta(x, y) = y/x \)). Derive a formula for the area of \( S \) in terms of an integral over the rectangle \([a, b] \times [\alpha, \beta]\) involving the function \( F \) and its partial derivatives \( F_r, F_\theta \).

Consider the vectorfield \( \vec{v}(x, y, z) = y\vec{i} - x\vec{j} \).

(i) Compute \( \text{curl} \ \vec{v} \).

(ii) Compute \( \oint_C \vec{v} \cdot \vec{t} \, ds \) where \( C \) is the unit circle in the \( xy \) plane, with counterclockwise orientation (seen from “above”).

(iii) If \( T \) is the triangle with corners \( P = (10, 0, 0) \), \( Q = (-5, 10, 0) \) and \( (5, 10, 0) \), then what is \( \int_T \vec{v} \cdot \vec{t} \, ds \)?

(iv) If \( T \) is the triangle with corners \( P = (10, 0, 0) \), \( Q = (-5, 10, 1) \) and \( (5, 10, -3) \), then what is \( \int_T \vec{v} \cdot \vec{t} \, ds \)?

(i) True or false: If \( \vec{v}(x, y, z) = \nabla f(x, y, z) \) for some function with continuous second derivatives, then \( \text{curl} \ \vec{v} = \vec{0} \).

(ii) True or false: If some continuously differentiable vector field \( \vec{v}(x, y, z) \) defined on a domain \( \mathcal{D} \subset \mathbb{R}^3 \) satisfies \( \text{curl} \ \vec{v} = \vec{0} \), then there is a function \( f : \mathcal{D} \to \mathbb{R} \) such that \( \vec{v}(x, y, z) = \nabla f(x, y, z) \).

Prove Green’s Theorem (in the plane, from the 1st midterm) assuming the domain is a rectangle \([a, b] \times [c, d]\).

State the transformation formula for double integrals. Give a proof when the integrand is \( f(x, y) \equiv 1 \).