FINAL PROJECTS
Math 415–Spring 2008

Rules

Work with one or two students per group. Choose one of the following projects. Hand in a draft version on Friday May 9. Hand in the final version on the date of the final exam. You can use the book, and a computer if you wish.

1 Traveling waves for the Fitzhugh–Nagumo and Fisher–Kolmogorov problems

The Fitzhugh–Nagumo equation for the potential along a nerve axon, and the Fisher–Kolmogorov equation for the spread of an advantageous gene from ecology both are partial differential equations of the form

\[
\frac{\partial u}{\partial t} = f(u) + \frac{\partial^2 u}{\partial x^2}
\]  

(1)

where one has two possible choices for the function \( f(u) \):

Fitzhugh–Nagumo : \( f(u) = u(1-u)(u-a) \quad 0 < a < 1 \) is a parameter.

Fisher–Kolmogorov : \( f(u) = u(1-u) \).

Here one looks for travelling wave solutions, i.e. solutions of the form \( u(x,t) = U(x - ct) \), where \( c \) is the “wave speed,” which satisfy

\[
\lim_{z \to \infty} U(z) = 1, \quad \lim_{z \to -\infty} U(z) = 0.
\]  

(2)

The function of one variable \( U(z) \) must then satisfy

\[
U''(z) + cu'(z) + f(U) = 0.
\]  

(3)

Note about exact solutions: For the Fitzhugh–Nagumo equation the exact solution is known to be

\[
U(z) = \frac{1}{1 + e^{z/\sqrt{2}}}, \quad c = (a - \frac{1}{2})\sqrt{2}.
\]  

(*)

The point of this problem is this: it is relatively straightforward to check that the function \( U \) above is indeed a solution of the Fitzhugh–Nagumo equation, but that doesn’t tell you that there aren’t any other solutions. Thus, knowing the solution (\( * \)) you don’t know if all traveling waves proceed at the same speed, or if there can be waves at a different wave speed \( c \).
For the Fitzhugh–Nagumo equation it turns out that there is only one traveling wave solution, but for the Fisher–Kolmogorov equation there is a whole family of traveling wave solutions, and only one of those can be written as an explicit formula like (*). By analyzing the phase plane that goes with the traveling wave equation (3) you can understand why for some equations there is only one traveling wave speed, while for another there are many different possible speeds.

1. Derive the second order ordinary differential equation (3) for \( U(z) \), and set up a system of first order equations for \( U, V = U' \).

2. Regarding the wave speed \( c \) as a parameter, find all fixed points and classify them according to their linearizations, for all values of \( c \).

3. Show that for \( c = 0 \) the system is conservative.

4. Show that any traveling wave corresponds to a heteroclinic trajectory from one fixed point to another.

5. Show for the Fitzhugh–Nagumo equation that there is exactly one value of \( c \) for which a traveling wave solution exists.

2 Hopf-bifurcation and Averaging

We consider two systems of differential equations with a parameter \( \alpha \):

\[
\begin{align*}
\dot{x} &= rx - \omega y - 2x^2y, & \dot{y} &= 2ry + \omega x \quad (4)
\end{align*}
\]

and

\[
\begin{align*}
\dot{x} &= rx - \omega y - 2xy^2, & \dot{y} &= 2ry + \omega x \quad (5)
\end{align*}
\]

This system has the origin as a fixed point for all parameter values \( \alpha \). The quantity \( \omega \) is a positive constant.

0. Show that the system (4) has a conserved quantity when \( r = 0 \). Show that this quantity is strictly decreasing when \( r > 0 \) (or \( r < 0 \)).

1. Find the linearization at the origin, determine its eigenvalues, and show that there is a Hopf-bifurcation at \( r = 0 \). (Both systems have the same linearization at the origin).

The goal of this project is to find the family of periodic orbits which emanates from the origin as the parameter \( r \) crosses \( r = 0 \). The rest of the questions are about the system (5).

Substitute \( x = \sqrt{|r|}X, \ y = \sqrt{|r|}Y \), and write \((X,Y)\) in polar coordinates

\[
X = R \cos \theta, \quad Y = R \sin \theta.
\]
2. Find the system of equations satisfied by \((X,Y)\). Be careful to distinguish between the cases \(r > 0\) and \(r < 0\).

3. Find an equation for \(\frac{dR}{d\theta}\).

4. Compute how much the radius \(R\) changes after an orbit “goes around once” (i.e. after \(\theta\) increases from \(\theta = 0\) to \(2\pi\)).

5. Show that for small positive or negative \(r\) the \((X,Y)\) system has a periodic orbit and determine its radius \(R_\ast\).

6. What does this say about the Hopf bifurcation in the original system (4)?

3 Turing instability

Alan Turing suggested that chemicals can react and diffuse so as to spatial patterns in such a way that they produce spatial patterns.

Consider the “Schnakenberg system” in which two chemical A and B react according to

\[
2A + B \rightarrow k_1 3A, \quad A \rightarrow k_2 \ast, \quad \ast \rightarrow k_3 A, \quad \ast \rightarrow k_4 B
\]

so the corresponding equations (by mass-action kinetics) are

\[
\begin{align*}
\dot{A} &= k_3 - k_2 A + k_1 A^2 B \\
\dot{B} &= k_4 - k_1 A^2 B
\end{align*}
\]

1. Show that by non-dimensionalization you can reduce the system to

\[
\begin{aligned}
\dot{u} &= \alpha - u + u^2 v \\
\dot{v} &= \beta - u^2 v
\end{aligned}
\]

2. Show that there is exactly one fixed point, for any value of the parameter \(\alpha > 0\).

3. Let \((U_\alpha, V_\alpha)\) be the coordinates of the fixed point. Show by linearizing that the fixed point is always stable.

Turing now constructed the following situation. The reactor is split into two components by a semipermeable barrier. The B-substance diffuses rapidly through the barrier, and so its concentration on both sides is the same \((v\) after non-dimensionalization), but A cannot cross the barrier, so the
concentrations $u_1$ and $u_2$ in both compartments are unrelated. One then gets the following system of equations for $(u_1, u_2, v)$

$$\begin{align*}
\dot{u}_1 &= \alpha - u_1 + u_1^2v \\
\dot{u}_2 &= \alpha - u_2 + u_2^2v \\
\dot{v} &= 1 - \left(\frac{u_1 + u_2}{2}\right)^2v
\end{align*}$$

(7)

4. Show that $u_1 = U\alpha, u_2 = U\alpha, v = V\alpha$ is a fixed point for this system.

5. Find the linearization at the fixed point $(U\alpha, U\alpha, V\alpha)$, and find the eigenvalues.

6. Find the eigenvector corresponding to the most unstable eigenvalue, i.e. the eigenvalue with the largest real part.

7. Interpret the result of your computation, in particular, can you find parameter values where

(a) the system with only one compartment (i.e. (6)) has a stable fixed point $(U\alpha, V\alpha)$, while

(b) the same equilibrium becomes unstable when the system is split into two compartments.

4 Activator–inhibitor model

(for “groups of one” only)

Consider the system given by

$$\begin{align*}
\dot{u} &= K\left\{A - u + \frac{H(u)}{v}\right\} \\
\dot{v} &= u^2 - v
\end{align*}$$

where $A$ and $K$ are positive parameters and where

$$H(u) = \frac{u^m}{1 + u^m}$$

is a Hill function, and $m$ is its “Hill coefficient.”

0. Assuming $m = 2$ draw the null-clines for the system.

1. Assuming $m = 2$ determine how many fixed points there are. Form a bifurcation diagram. Which bifurcations occur?
[as observed in class, you can parametrize the fixed points by their \( u \) coordinate, i.e. you can write them as \((u, v(u), A(u))\). Explain how the graph of \( A \) as a function of \( u \) gives you a bifurcation diagram.]

2. Determine the linearized stability of the fixed points.

3. Find Hopf-bifurcations which occur as one varies the parameter \( A \) from 0 to \( \infty \).

4. Show that for large values of \( K \) you get a “fast-slow” system, and use this knowledge to find periodic solutions for large \( K \).