PROBLEM SET I

Math 421; February 26, 2003

1. INDUCTION PROOFS

1. Show that for all \( n = 1, 2, 3, \ldots \) and \( x > -1 \) one has \((1 + x)^n \geq 1 + nx\).

2. Show by induction that for all \( n = 1, 2, 3, \ldots \) one has
   \[1 + 3 + 5 + \cdots + (2n - 1) = n^2.\]

3. There is a constant \( C \) such that for all \( n = 1, 2, 3, \ldots \) one has
   \[\frac{1}{1 \cdot 2} +\frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = C - \frac{1}{n+1}\]
   Find the constant \( C \), and prove the formula by induction on \( n \).

4. Find a formula for
   \[\frac{1}{1 \cdot 2 \cdot 3} +\frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1) \cdot (n+2)}\]
   and prove it by induction on \( n \).

5. Prove by induction that
   \[\prod_{k=1}^{n} \frac{k}{k+1} = \frac{2}{(n+1)(n+2)}\] for all \( n \geq 2 \).

2. ABOUT CONTINUOUS FUNCTIONS

1. Prove: If a function \( f \) satisfies \( f(x)^2 \leq |x| \) for all \( x \in [-1, 1] \) then \( f \) is continuous at \( x = 0 \).
   Must \( f \) be continuous at any other point in the interval \([-1, 1]\)?

2. Suppose a function \( f \) satisfies \( f(x+y) = f(x) + f(y) \) for all numbers \( x \) and \( y \).
   Suppose in addition that \( f \) is continuous at \( x = 0 \). Show that \( f \) is continuous at \( x = 1 \).

3. Suppose a function \( f \) is continuous at \( x = 2 \), and that \( f(2) = 0 \). Let \( \alpha \) be any positive number. Show that there is an open interval \((a, b)\) containing \( 2 \) such that \( f(x) + \alpha > 0 \) for all \( x \in (a, b) \).

4. Let \( f \) be a function which is known to be continuous on some interval \((a, b)\).
   Show that the function \( g(x) = |f(x)| \) and \( h(x) = f(-x) \) are also continuous.
   (Be precise: Where are \( g \) and \( h \) continuous?)

3. BOUNDED FUNCTIONS

Definition: A function \( f \) on an interval \((a, b)\) is said to be bounded if there exists a number \( M \) such that \( |f(x)| \leq M \) for all \( x \) in the interval \((a, b)\).

1. Show that the function \( f(x) = 1/x \) is not bounded on the interval \((0,1)\).

2. Is the function \( f(x) = 1/(1 + x^2) \) bounded on the interval \((\infty, \infty)\)?

3. A function \( f \) which is defined on the interval \([0, 1]\) is known to be bounded on \([0, \frac{2}{3}]\) and on \([\frac{1}{2}, 1]\). Must \( f \) be bounded on the interval \([0, 1]\)?

4. Suppose a function \( f \) is defined on the interval \((0, 1)\), and suppose that for every \( \varepsilon > 0 \) this function is known to be bounded on the interval \((\varepsilon, 1}\). Does it follow that \( f \) is bounded on the interval \((0, 1)\)?