Complex Numbers and the Complex Exponential

1. Complex numbers

The equation $x^2 + 1 = 0$ has no solutions, because for any real number x the square x^2 is nonnegative, and so $x^2 + 1$ can never be less than 1. In spite of this it turns out to be very useful to ***assume*** that there is a number i for which one has

$$i^2 = -1. \tag{1}$$

Any ***complex number*** is then an expression of the form $a + bi$, where a and b are old-fashioned real numbers. The number a is called the ***real part*** of $a + bi$, and b is called its ***imaginary part.***

Traditionally the letters z and w are used to stand for complex numbers.

Since any complex number is specified by two real numbers one can visualize them by plotting a point with coordinates (a, b) in the plane for a complex number $a + bi$. The plane in which one plot these complex numbers is called the Complex plane, or Argand plane.

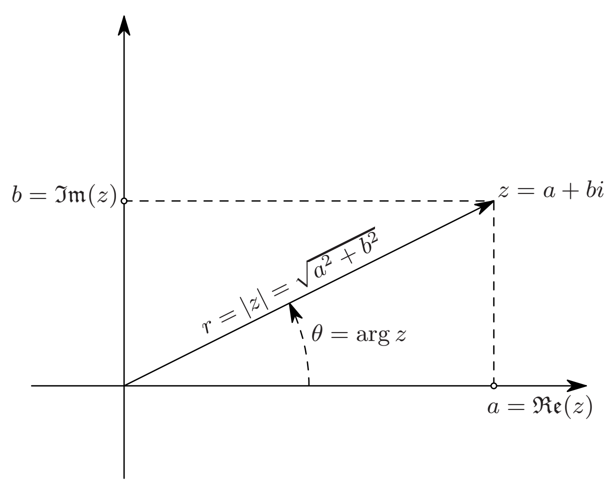


Figure 1. A complex number.

You can add, multiply and divide complex numbers. Here's how:

To add (subtract) $z = a + bi$ and $w = c + di$

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$z - w = (a + bi) - (c + di) = (a - c) + (b - d)i.$$

To multiply z and w proceed as follows:

$$\begin{aligned} zw &= (a+bi)(c+di) \\ &= a(c+di) + bi(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac-bd) + (ad+bc)i \end{aligned}$$

where we have use the defining property $i^2 = -1$ to get rid of i^2.

To divide two complex numbers one always uses the following trick.

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} \\ &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \end{aligned}$$

Now

$$(c+di)(c-di) = c^2 - (di)^2 = c^2 - d^2 i^2 = c^2 + d^2,$$

so

$$\begin{aligned} \frac{a+bi}{c+di} &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i \end{aligned}$$

Obviously you do not want to memorize this formula: instead you remember the trick, i.e. to divide $c + di$ into $a + bi$ you multiply numerator and denominator with $c - di$.

For any complex number $w = c+di$ the number $c-di$ is called its ***complex conjugate.*** Notation:

$$w = c + di, \quad \bar{w} = c - di.$$

A frequently used property of the complex conjugate is the following formula

$$w\bar{w} = (c+di)(c-di) = c^2 - (di)^2 = c^2 + d^2. \tag{2}$$

The following notation is used for the ***real and imaginary parts*** of a complex number z. If $z = a + bi$ then

$$a = \text{the Real Part of } z = \mathfrak{Re}(z), \qquad b = \text{the Imaginary Part of } z = \mathfrak{Im}(z).$$

Note that both $\mathfrak{Re} z$ and $\mathfrak{Im} z$ are real numbers. A common mistake is to say that $\mathfrak{Im} z = bi$. The "i" should **not** be there.

2. Argument and Absolute Value

For any given complex number $z = a+bi$ one defines the ***absolute value*** or ***modulus*** to be

$$|z| = \sqrt{a^2 + b^2},$$

so $|z|$ is the distance from the origin to the point z in the complex plane (see figure 1).

The angle θ is called the ***argument*** of the complex number z. Notation:

$$\arg z = \theta.$$

The argument is defined in an ambiguous way: it is only defined up to a multiple of 2π. E.g. the argument of -1 could be π, or $-\pi$, or 3π, or, etc. In general one says $\arg(-1) = \pi + 2k\pi$, where k may be any integer.

From trigonometry one sees that for any complex number $z = a + bi$ one has

$$a = |z| \cos\theta, \text{ and } b = |z| \sin\theta,$$

so that

$$|z| = |z|\cos\theta + i|z|\sin\theta = |z|\bigl(\cos\theta + i\sin\theta\bigr).$$

and

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$

2.1. Example: Find argument and absolute value of $z = 2 + i$. *Solution:* $|z| = \sqrt{2^2 + 1^2} = \sqrt{5}$. z lies in the first quadrant so its argument θ is an angle between 0 and $\pi/2$. From $\tan\theta = \frac{1}{2}$ we then conclude $\arg(2 + i) = \theta = \arctan\frac{1}{2}$.

3. Geometry of Arithmetic

Since we can picture complex numbers as points in the complex plane, we can also try to visualize the arithmetic operations "addition" and "multiplication." To add z and

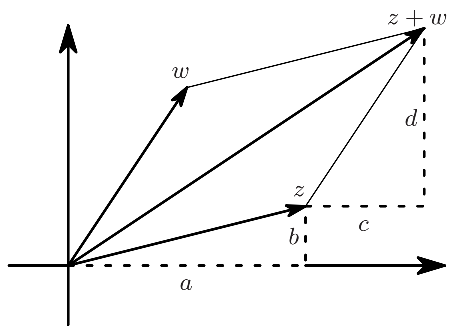


Figure 2. Addition of $z = a + bi$ and $w = c + di$

w one forms the parallelogram with the origin, z and w as vertices. The fourth vertex then is $z + w$. See figure 2.

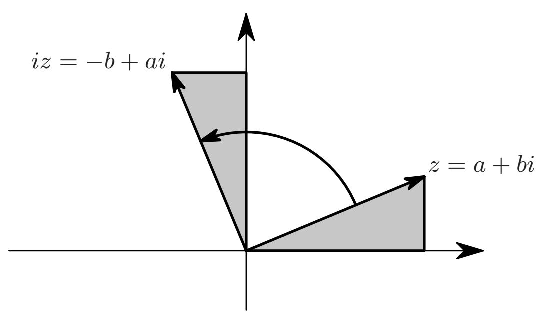


Figure 3. Multiplication of $a + bi$ by i.

To understand multiplication we first look at multiplication with i. If $z = a + bi$ then

$$iz = i(a + bi) = ia + bi^2 = ai - b = -b + ai.$$

Thus, to form iz from the complex number z one rotates z counterclockwise by 90 degrees. See figure 3.

If a is any real number, then multiplication of $w = c + di$ by a gives

$$aw = ac + adi,$$

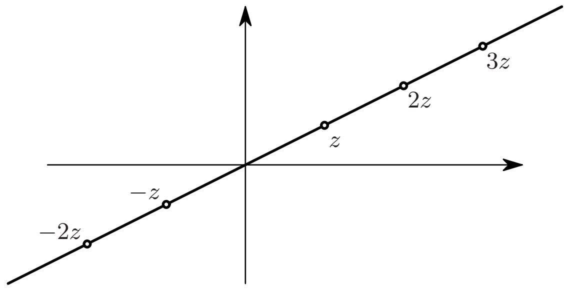


Figure 4. Multiplication of a real and a complex number

so aw points in the same direction, but is a times as far away from the origin. If $a < 0$ then aw points in the opposite direction. See figure 4.

Next, to multiply $z = a + bi$ and $w = c + di$ we write the product as

$$zw = (a + bi)w = aw + biw.$$

Figure 5 shows $a + bi$ on the right. On the left, the complex number w was first drawn,

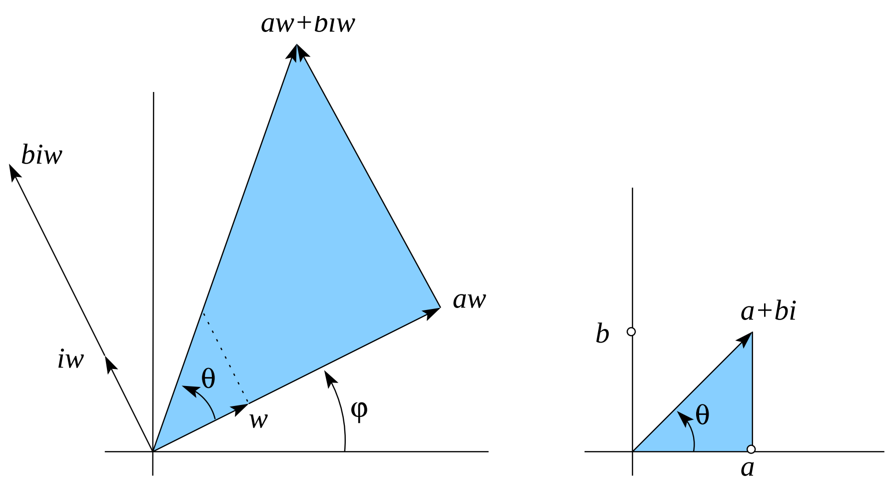


Figure 5. Multiplication of two complex numbers

then aw was drawn. Subsequently iw and biw were constructed, and finally $zw = aw+biw$ was drawn by adding aw and biw.

One sees from figure 5 that since iw is perpendicular to w, the line segment from 0 to biw is perpendicular to the segment from 0 to aw. Therefore the larger shaded triangle on the left is a right triangle. The length of the adjacent side is $a|w|$, and the length of the opposite side is $b|w|$. The ratio of these two lengths is $a : b$, which is the same as for the shaded right triangle on the right, so we conclude that these two triangles are similar.

The triangle on the left is $|w|$ times as large as the triangle on the right. The two angles marked θ are equal.

Since $|zw|$ is the length of the hypothenuse of the shaded triangle on the left, it is $|w|$ times the hypothenuse of the triangle on the right, i.e. $|zw| = |w| \cdot |z|$.

The argument of zw is the angle $\theta+\varphi$; since $\theta = \arg z$ and $\varphi = \arg w$ we get the following two formulas

$$|zw| = |z| \cdot |w| \tag{3}$$

$$\arg(zw) = \arg z + \arg w, \tag{4}$$

in other words,

when you multiply complex numbers, their lengths get multiplied and their arguments get added.

4. Applications in Trigonometry

4.1. Unit length complex numbers. For any θ the number $z = \cos\theta + i\sin\theta$ has length 1: it lies on the unit circle. Its argument is $\arg z = \theta$. Conversely, any complex number on the unit circle is of the form $\cos\phi + i\sin\phi$, where ϕ is its argument.

4.2. The Addition Formulas for Sine & Cosine. For any two angles θ and ϕ one can multiply $z = \cos\theta + i\sin\theta$ and $w = \cos\phi + i\sin\phi$. The product zw is a complex number of absolute value $|zw| = |z|\cdot|w| = 1\cdot 1$, and with argument $\arg(zw) = \arg z + \arg w = \theta + \phi$. So zw lies on the unit circle and must be $\cos(\theta+\phi) + i\sin(\theta+\phi)$. Thus we have

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi). \tag{5}$$

By multiplying out the Left Hand Side we get

$$\begin{aligned}(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi \\ &\quad + i(\sin\theta\cos\phi + \cos\theta\sin\phi).\end{aligned} \tag{6}$$

Compare the Right Hand Sides of (5) and (6), and you get the addition formulas for Sine and Cosine:

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

4.3. De Moivre's formula. For any complex number z the argument of its square z^2 is $\arg(z^2) = \arg(z\cdot z) = \arg z + \arg z = 2\arg z$. The argument of its cube is $\arg z^3 = \arg(z\cdot z^2) = \arg(z) + \arg z^2 = \arg z + 2\arg z = 3\arg z$. Continuing like this one finds that

$$\arg z^n = n\ \arg z \tag{7}$$

for any integer n.

Applying this to $z = \cos\theta + i\sin\theta$ you find that z^n is a number with absolute value $|z^n| = |z|^n = 1^n = 1$, and argument $n\arg z = n\theta$. Hence $z^n = \cos n\theta + i\sin n\theta$. So we have found

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{8}$$

This is ***de Moivre's formula.***

For instance, for $n = 2$ this tells us that

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$$

Comparing real and imaginary parts on left and right hand sides this gives you the double angle formulas $\cos\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

For $n = 3$ you get, using the *Binomial Theorem,* or Pascal's triangle,

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

so that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and

$$\sin 3\theta = \cos^2\theta\sin\theta - \sin^3\theta.$$

In this way it is fairly easy to write down similar formulas for $\sin 4\theta$, $\sin 5\theta$, etc.. . .

5. Calculus of complex valued functions

A ***complex valued function*** on some interval $I = (a, b) \subseteq \mathbb{R}$ is a function $f : I \to \mathbb{C}$. Such a function can be written as in terms of its real and imaginary parts,

$$f(x) = u(x) + iv(x), \tag{9}$$

in which $u, v : I \to \mathbb{R}$ are two real valued functions.

One defines limits of complex valued functions in terms of limits of their real and imaginary parts. Thus we say that

$$\lim_{x\to x_0} f(x) = L$$

if $f(x) = u(x) + iv(x)$, $L = A + iB$, and both

$$\lim_{x\to x_0} u(x) = A \text{ and } \lim_{x\to x_1} v(x) = B$$

hold. From this definition one can prove that the usual limit theorems also apply to complex valued functions.

5.1. Theorem. *If* $\lim_{x\to x_0} f(x) = L$ *and* $\lim_{x\to x_0} g(x) = M$*, then one has*

$$\lim_{x\to x_0} f(x) \pm g(x) = L \pm M,$$

$$\lim_{x\to x_0} f(x)g(x) = LM,$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}, \text{ provided } M \neq 0.$$

The ***derivative*** of a complex valued function $f(x) = u(x)+iv(x)$ is defined by simply differentiating its real and imaginary parts:

$$f'(x) = u'(x) + iv'(x). \tag{10}$$

Again, one finds that the sum,product and quotient rules also hold for complex valued functions.

5.2. Theorem. *If* $f, g : I \to \mathbb{C}$ *are complex valued functions which are differentiable at some* $x_0 \in I$*, then the functions* $f \pm g$*,* fg *and* f/g *are differentiable (assuming* $g(x_0) \neq 0$ *in the case of the quotient.) One has*

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Note that the chain rule does not appear in this list! See problem 29 for more about the chain rule.

6. The Complex Exponential Function

We finally give a definition of e^{a+bi}. First we consider the case $a = 0$:

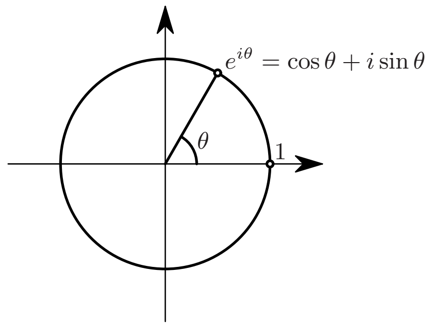


Figure 6. Euler's definition of $e^{i\theta}$

6.1. Definition. *For any real number t we set*

$$e^{it} = \cos t + i \sin t.$$

See Figure 6.

6.2. Example. $e^{\pi i} = \cos \pi + i \sin \pi = -1$. This leads to Euler's famous formula

$$e^{\pi i} + 1 = 0,$$

which combines the five most basic quantities in mathematics: e, π, i, 1, and 0.

Reasons why the definition 6.1 seems a good definition.

Reason 1. We haven't defined e^{it} before and we can do anything we like.

Reason 2. Substitute it in the Taylor series for e^x:

$$\begin{aligned} e^{it} &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots \\ &= 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} - \cdots \\ &= 1 - t^2/2! + t^4/4! - \cdots \\ &\qquad + i\big(t - t^3/3! + t^5/5! - \cdots\big) \\ &= \cos t + i \sin t. \end{aligned}$$

This is not a proof, because before we had only proved the convergence of the Taylor series for e^x if x was a real number, and here we have pretended that the series is also good if you substitute $x = it$.

Reason 3. As a function of t the definition 6.1 gives us the correct derivative. Namely, using the chain rule (i.e. pretending it still applies for complex functions) we would get

$$\frac{de^{it}}{dt} = ie^{it}.$$

Indeed, this is correct. To see this proceed from our definition 6.1:

$$\begin{aligned} \frac{de^{it}}{dt} &= \frac{d \cos t + i \sin t}{dt} \\ &= \frac{d\cos t}{dt} + i\frac{d \sin t}{dt} \\ &= -\sin t + i \cos t \\ &= i(\cos t + i \sin t) \end{aligned}$$

Reason 4. The formula $e^x \cdot e^y = e^{x+y}$ still holds. Rather, we have $e^{it+is} = e^{it}e^{is}$. To check this replace the exponentials by their definition:

$$e^{it}e^{is} = (\cos t + i\sin t)(\cos s + i\sin s) = \cos(t+s) + i\sin(t+s) = e^{i(t+s)}.$$

Requiring $e^x \cdot e^y = e^{x+y}$ to be true for all complex numbers helps us decide what e^{a+bi} shoud be for arbitrary complex numbers $a+bi$.

6.3. Definition. *For any complex number $a+bi$ we set*

$$e^{a+bi} = e^a \cdot e^{ib} = e^a(\cos b + i\sin b).$$

One verifies as above in "reason 3" that this gives us the right behaviour under differentiation. Thus, for any complex number $r = a+bi$ the function

$$y(t) = e^{rt} = e^{at}(\cos bt + i\sin bt)$$

satisfies

$$y'(t) = \frac{de^{rt}}{dt} = re^{rt}.$$

7. Complex solutions of polynomial equations

7.1. Quadratic equations. The well-known quadratic formula tells you that the equation

$$ax^2 + bx + c = 0 \tag{11}$$

has two solutions, given by

$$x_\pm = \frac{-b \pm \sqrt{D}}{2a}, \qquad D = b^2 - 4ac. \tag{12}$$

If the coefficients a, b, c are real numbers and if the ***discriminant*** D is positive, then this formula does indeed give two real solutions x_+ and x_-. However, if $D < 0$, then there are no real solutions, but there are two complex solutions, namely

$$x_\pm = \frac{-b}{2a} \pm i\frac{\sqrt{-D}}{2a}$$

7.2. Example: solve $x^2 + 2x + 5 = 0$. *Solution:* Use the quadratic formula, or complete the square:

$$\begin{aligned} & x^2 + 2x + 5 = 0 \\ \iff & x^2 + 2x + 1 = -4 \\ \iff & (x+1)^2 = -4 \\ \iff & x + 1 = \pm 2i \\ \iff & x = -1 \pm 2i. \end{aligned}$$

So, if you allow complex solutions then every quadratic equation has two solutions, unless the two solutions coincide (the case $D = 0$, in which there is only one solution.)

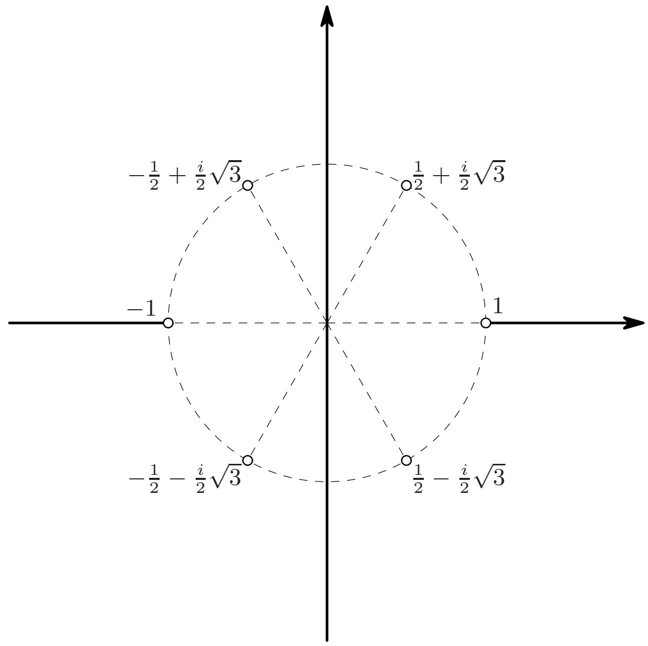


Figure 7. The sixth roots of 1. There are six of them, and they re arranged in a regular hexagon.

7.3. Complex roots of a number. For any given complex number w there is a method of finding all complex solutions of the equation

$$z^n = w \tag{13}$$

if $n = 2, 3, 4, \cdots$ is a given integer.

To find these solutions you write w in polar form, i.e. you find $r > 0$ and θ such that $w = re^{i\theta}$. Then

$$z = r^{1/n}e^{i\theta/n}$$

is a solution to (13). But it isn't the only solution, because the angle θ for which $w = r^{i\theta}$ isn't unique – it is only determined up to a multiple of 2π. Thus if we have found one angle θ for which $w = r^{i\theta}$, then we can also write

$$w = re^{i(\theta+2k\pi)}, \qquad k = 0, \pm1, \pm2, \cdots$$

The n^{th} roots of w are then

$$z_k = r^{1/n}e^{i\left(\frac{\theta}{n}+2\frac{k}{n}\pi\right)}$$

Here k can be any integer, so it looks as if there are infinitely many solutions. However, if you increase k by n, then the exponent above increases by $2\pi i$, and hence z_k does not change. In a formula:

$$z_n = z_0, \quad z_{n+1} = z_1, \quad z_{n+2} = z_2, \quad \ldots \quad z_{k+n} = z_k$$

So if you take $k = 0, 1, 2, \cdots, n-1$ then you have had all the solutions.

The solutions z_k always form a regular polygon with n sides.

7.4. Example: find all sixth roots of $w = 1$. We are to solve $z^6 = 1$. First write 1 in polar form,

$$1 = 1 \cdot e^{0i} = 1 \cdot e^{2k\pi i}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Then we take the 6^{th} root and find

$$z_k = 1^{1/6} e^{2k\pi i/6} = e^{k\pi i/3}, \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The six roots are

$$z_0 = 1 \qquad z_1 = e^{\pi i/3} = \tfrac{1}{2} + \tfrac{i}{2}\sqrt{3} \qquad z_2 = e^{2\pi i/3} = -\tfrac{1}{2} + \tfrac{i}{2}\sqrt{3}$$

$$z_3 = -1 \qquad z_4 = e^{\pi i/3} = -\tfrac{1}{2} - \tfrac{i}{2}\sqrt{3} \qquad z_5 = e^{\pi i/3} = \tfrac{1}{2} - \tfrac{i}{2}\sqrt{3}$$

8. Other handy things you can do with complex numbers

8.1. Partial fractions. Consider the partial fraction decomposition

$$\frac{x^2 + 3x - 4}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 4}$$

The coefficient A is easy to find: multiply with $x - 2$ and set $x = 2$ (or rather, take the limit $x \to 2$) to get

$$A = \frac{2^2 + 3 \cdot 2 - 4}{2^2 + 4} = \cdots.$$

Before we had no similar way of finding B and C quickly, but now we can apply the same trick: multiply with $x^2 + 4$,

$$\frac{x^2 + 3x - 4}{(x-2)} = Bx + C + (x^2 + 4)\frac{A}{x-2},$$

and substitute $x = 2i$. This make $x^2 + 4 = 0$, with result

$$\frac{(2i)^2 + 3 \cdot 2i - 4}{(2i - 2)} = 2iB + C.$$

Simplify the complex number on the left:

$$\begin{aligned}
\frac{(2i)^2 + 3 \cdot 2i - 4}{(2i - 2)} &= \frac{-4 + 6i - 4}{-2 + 2i} \\
&= \frac{-8 + 6i}{-2 + 2i} \\
&= \frac{(-8 + 6i)(-2 - 2i)}{(-2)^2 + 2^2} \\
&= \frac{28 + 4i}{8} \\
&= \frac{7}{2} + \frac{i}{2}
\end{aligned}$$

So we get $2iB + C = \frac{7}{2} + \frac{i}{2}$; since B and C are real numbers this implies

$$B = \frac{1}{4}, \qquad C = \frac{7}{2}.$$

8.2. Certain trigonometric and exponential integrals. You can compute

$$I = \int e^{3x} \cos 2x \mathrm{d}x$$

by integrating by parts twice. You can also use that $\cos 2x$ is the real part of e^{2ix}. Instead of computing the real integral I, we look at the following related complex integral

$$J = \int e^{3x} e^{2ix} \mathrm{d}x$$

which we get from I by replacing $\cos 2x$ with e^{2ix}. Since $e^{2ix} = \cos 2x + i \sin 2x$ we have

$$J = \int e^{3x}(\cos 2x + i \sin 2x)\mathrm{d}x = \int e^{3x} \cos 2x \mathrm{d}x + i \int e^{3x} \sin 2x \mathrm{d}x$$

i.e.,

$$J = I + \text{something imaginary.}$$

The point of all this is that J is easier to compute than I:

$$J = \int e^{3x} e^{2ix} \mathrm{d}x = \int e^{3x+2ix} \mathrm{d}x = \int e^{(3+2i)x} \mathrm{d}x = \frac{e^{(3+2i)x}}{3+2i} + C$$

where we have used that

$$\int e^{ax} \mathrm{d}x = \frac{1}{a} e^{ax} + C$$

holds even if a is complex is a complex number such as $a = 3 + 2i$.

To find I you have to compute the real part of J, which you do as follows:

$$\begin{aligned}\frac{e^{(3+2i)x}}{3+2i} &= e^{3x} \frac{\cos 2x + i \sin 2x}{3+2i} \\ &= e^{3x} \frac{(\cos 2x + i \sin 2x)(3-2i)}{(3+2i)(3-2i)} \\ &= e^{3x} \frac{3 \cos 2x + 2 \sin 2x + i(\cdots)}{13}\end{aligned}$$

so

$$\int e^{3x} \cos 2x \mathrm{d}x = e^{3x} \left(\tfrac{3}{13} \cos 2x + \tfrac{2}{13} \sin 2x\right) + C.$$

8.3. Complex amplitudes. A harmonic oscillation is given by

$$y(t) = A \cos(\omega t - \phi),$$

where A is the ***amplitude***, ω is the ***frequency***, and ϕ is the ***phase*** of the oscillation. If you add two harmonic oscillations with the same frequency ω, then you get another harmonic oscillation with frequency ω. You can prove this using the addition formulas for cosines, but there's another way using complex exponentials. It goes like this.

Let $y(t) = A \cos(\omega t - \phi)$ and $z(t) = B \cos(\omega t - \theta)$ be the two harmonic oscillations we wish to add. They are the real parts of

$$\begin{aligned}Y(t) &= A \{\cos(\omega t - \phi) + i \sin(\omega t - \phi)\} = A e^{i\omega t - i\phi} = A e^{-i\phi} e^{i\omega t} \\ Z(t) &= B \{\cos(\omega t - \theta) + i \sin(\omega t - \theta)\} = B e^{i\omega t - i\theta} = B e^{-i\theta} e^{i\omega t}\end{aligned}$$

Therefore $y(t) + z(t)$ is the real part of $Y(t) + Z(t)$, i.e.

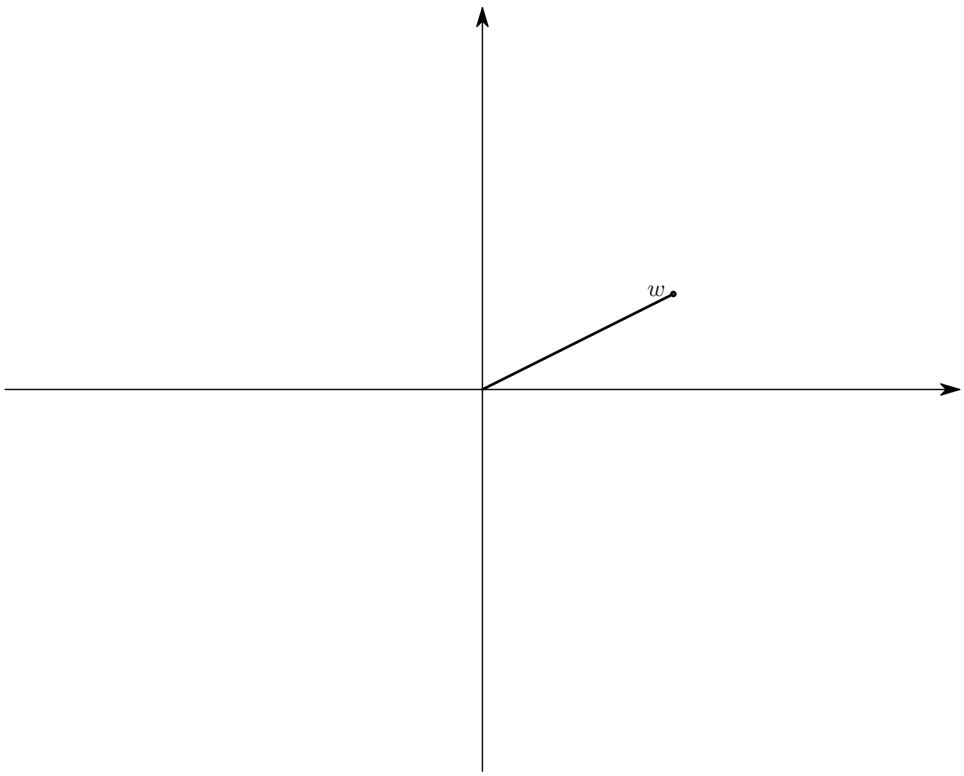
$$y(t) + z(t) = \mathfrak{Re}\big(Y(t)\big) + \mathfrak{Re}\big(Z(t)\big) = \mathfrak{Re}\big(Y(t) + Z(t)\big).$$

The quantity $Y(t) + Z(t)$ is easy to compute:

$$Y(t) + Z(t) = A e^{-i\phi} e^{i\omega t} + B e^{-i\theta} e^{i\omega t} = \left(A e^{-i\phi} + B e^{-i\theta}\right) e^{i\omega t}.$$

If you now do the complex addition

$$A e^{-i\phi} + B e^{-i\theta} = C e^{-i\psi},$$



i.e. you add the numbers on the right, and compute the absolute value C and argument $-\psi$ of the sum, then we see that $Y(t) + Z(t) = Ce^{i(\omega t - \psi)}$. Since we were looking for the real part of $Y(t) + Z(t)$, we get

$$y(t) + z(t) = A\cos(\omega t - \phi) + B\cos(\omega t - \theta) = C\cos(\omega t - \psi).$$

The complex numbers $Ae^{-i\phi}$, $Be^{-i\theta}$ and $Ce^{-i\psi}$ are called the complex amplitudes for the harmonic oscillations $y(t)$, $z(t)$ and $y(t) + z(t)$.

The recipe for adding harmonic oscillations can therefore be summarized as follows: ***Add the complex amplitudes.***

9. PROBLEMS

Computing and Drawing Complex Numbers.

1. Compute the following complex numbers by hand.

Draw ***all*** numbers in the complex (or "Argand") plane (use graph paper or quad paper if necessary).

Compute absolute value and argument of all numbers involved.

i^2; i^3; i^4; $1/i$;

$(1 + 2i)(2 - i)$;

$(1 + i)(1 + 2i)(1 + 3i)$;

$(\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2})^2$; $(\frac{1}{2} + \frac{i}{2}\sqrt{3})^3$;

$\dfrac{1}{1+i}$; $5/(2 - i)$;

2. [Deriving the addition formula for $\tan(\theta + \phi)$**]** Let $\theta, \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be two angles.

(a) What are the arguments of

$$z = 1 + i\tan\theta \text{ and } w = 1 + i\tan\phi?$$

(Draw both z and w.)

(b) Compute zw.

(c) What is the argument of zw?

(d) Compute $\tan(\arg zw)$.

3. Find formulas for $\cos 4\theta$, $\sin 4\theta$, $\cos 5\theta$ and $\sin 6\theta$ in terms of $\cos\theta$ and $\sin\theta$, by using *de Moivre*'s formula.

4. In the following picture draw $2w$, $\frac{3}{4}w$, iw, $-2iw$, $(2+i)w$ and $(2-i)w$. (Try to make a nice drawing, use a ruler.)

Make a new copy of the picture, and draw $\bar{w}$, $-\bar{w}$ and $-w$.

Make yet another copy of the drawing. Draw $1/w$, $1/\bar{w}$, and $-1/w$. For this drawing you need to know where the unit circle is in your drawing: Draw a circle centered at the origin with radius of your choice, and let this be the unit circle. [Depending on which circle you draw you will get a different answer!]

5. Verify directly from the definition of addition and multiplication of complex numbers that

(a) $z + w = w + z$

(b) $zw = wz$

(c) $z(v + w) = zv + zw$

holds for *all* complex numbers v, w, and z.

6. True or False? (In mathematics this means that you should either give a proof that the statement is always true, or else give a counterexample, thereby showing that the statement is not always true.)

For any complex numbers z and w one has

(a) $\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(z + w)$

(b) $\overline{z + w} = \bar{z} + \bar{w}$

(c) $\mathfrak{Im}(z) + \mathfrak{Im}(w) = \mathfrak{Im}(z + w)$

(d) $\overline{zw} = (\bar{z})(\bar{w})$

(e) $\mathfrak{Re}(z)\mathfrak{Re}(w) = \mathfrak{Re}(zw)$

(f) $\overline{z/w} = (\bar{z})/(\bar{w})$

(g) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$

(h) $\mathfrak{Re}(iz) = i\mathfrak{Re}(z)$

(i) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$

(j) $\mathfrak{Re}(iz) = i\mathfrak{Im}(z)$

(k) $\mathfrak{Im}(iz) = \mathfrak{Re}(z)$

(l) $\mathfrak{Re}(\bar{z}) = \mathfrak{Re}(z)$

7. The imaginary part of a complex number is known to be twice its real part. The absolute value of this number is 4. Which number is this?

8. The real part of a complex number is known to be half the absolute value of that number. The imaginary part of the number is 1. Which number is it?

The Complex Exponential.

9. Compute and **draw** the following numbers in the complex plane

$e^{\pi i/3}$; $e^{\pi i/2}$; $\sqrt{2}e^{3\pi i/4}$; $e^{17\pi i/4}$.

$e^{\pi i} + 1$; $e^{i\ln 2}$.

$\dfrac{1}{e^{\pi i/4}}$; $\dfrac{e^{-\pi i}}{e^{\pi i/4}}$; $\dfrac{e^{2-\pi i/2}}{e^{\pi i/4}}$

$e^{2009\pi i}$; $e^{2009\pi i/2}$.

$-8e^{4\pi i/3}$; $12e^{\pi i} + 3e^{-\pi i}$.

10. Compute the absolute value and argument of $e^{(\ln 2)(1+i)}$.

11. Suppose z can be any complex number.

(a) Is it true that e^z is always a positive number?

(b) Is it true that $e^z \neq 0$?

12. Verify directly from the definition that

$$e^{-it} = \frac{1}{e^{it}}$$

holds for *all real* values of t.

13. Show that

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

14. Show that

$$\cosh x = \cos ix, \quad \sinh x = \frac{1}{i}\sin ix.$$

15. The general solution of a second order linear differential equation contains expressions of the form $Ae^{i\beta t} + Be^{-i\beta t}$. These can be rewritten as $C_1\cos\beta t + C_2\sin\beta t$.

If $Ae^{i\beta t} + Be^{-i\beta t} = 2\cos\beta t + 3\sin\beta t$, then what are A and B?

16. (a) Show that you can write a "cosine-wave" with amplitude A and phase ϕ as follows

$$A\cos(t - \phi) = \mathfrak{Re}\left(ze^{it}\right),$$

where the "complex amplitude" is given by $z = Ae^{-i\phi}$. (See §8.3).

(b) Show that a "sine-wave" with amplitude A and phase ϕ as follows

$$A\sin(t-\phi) = \mathfrak{Re}\left(ze^{it}\right),$$

where the "complex amplitude" is given by $z = -iAe^{-i\phi}$.

17. Find A and ϕ where $A\cos(t-\phi) = 2\cos(t) + 2\cos(t - \frac{2}{3}\pi)$.

18. Find A and ϕ where $A\cos(t-\phi) = 12\cos(t - \frac{1}{6}\pi) + 12\sin(t - \frac{1}{3}\pi)$.

19. Find A and ϕ where $A\cos(t-\phi) = 12\cos(t-\pi/6) + 12\cos(t-\pi/3)$.

20. Find A and ϕ such that $A\cos(t-\phi) = \cos\left(t - \frac{1}{6}\pi\right) + \sqrt{3}\cos\left(t - \frac{2}{3}\pi\right)$.

Real and Complex Solutions of Algebraic Equations.

21. ***Find and draw*** all real and complex solutions of

(a) $z^2 + 6z + 10 = 0$

(b) $z^3 + 8 = 0$

(c) $z^3 - 125 = 0$

(d) $2z^2 + 4z + 4 = 0$

(e) $z^4 + 2z^2 - 3 = 0$

(f) $3z^6 = z^3 + 2$

(g) $z^5 - 32 = 0$

(h) $z^5 - 16z = 0$

Calculus of Complex Valued Functions.

22. Compute the derivatives of the following functions

$$f(x) = \frac{1}{x+i} \qquad g(x) = \log x + i\arctan x$$

$$h(x) = e^{ix^2} \qquad k(x) = \log\frac{i+x}{i-x}$$

Try to simplify your answers.

23. (a) Compute

$$\int \left(\cos 2x\right)^4 \, dx$$

by using $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and expanding the fourth power.

(b) Assuming $a \in \mathbb{R}$, compute

$$\int e^{-2x}\left(\sin ax\right)^2 \, dx.$$

(same trick: write $\sin ax$ in terms of complex exponentials; make sure your final answer has no complex numbers.)

24. Use $\cos\alpha = (e^{i\alpha} + e^{-i\alpha})/2$, etc. to evaluate these indefinite integrals:

(a) $\displaystyle\int \cos^2 x \, \mathrm{d}x$

(b) $\displaystyle\int \cos^4 x \, \mathrm{d}x,$

(c) $\displaystyle\int \cos^2 x \sin x \, \mathrm{d}x,$

(d) $\displaystyle\int \sin^3 x \, \mathrm{d}x,$

(e) $\displaystyle\int \cos^2 x \sin^2 x \, \mathrm{d}x,$

(f) $\displaystyle\int \sin^6 x \, \mathrm{d}x$

(g) $\displaystyle\int \sin(3x)\cos(5x) \, \mathrm{d}x$

(h) $\displaystyle\int \sin^2(2x)\cos(3x) \, \mathrm{d}x$

(i) $\displaystyle\int_0^{\pi/4} \sin(3x)\cos(x) \, \mathrm{d}x$

(j) $\displaystyle\int_0^{\pi/3} \sin^3(x)\cos^2(x) \, \mathrm{d}x$

(k) $\displaystyle\int_0^{\pi/2} \sin^2(x)\cos^2(x) \, \mathrm{d}x$

(l) $\displaystyle\int_0^{\pi/3} \sin(x)\cos^2(x) \, \mathrm{d}x$

25. Compute the following integrals when $m \neq n$ are distinct integers.

(a) $\displaystyle\int_0^{2\pi} \sin(mx)\cos(nx) \, \mathrm{d}x$

(b) $\displaystyle\int_0^{2\pi} \sin(nx)\cos(nx) \, \mathrm{d}x$

(c) $\displaystyle\int_0^{2\pi} \cos(mx)\cos(nx) \, \mathrm{d}x$

(d) $\displaystyle\int_0^{\pi} \cos(mx)\cos(nx) \, \mathrm{d}x$

(e) $\displaystyle\int_0^{2\pi} \sin(mx)\sin(nx) \, \mathrm{d}x$

(f) $\displaystyle\int_0^{\pi} \sin(mx)\sin(nx) \, \mathrm{d}x$

These integrals are basic to the theory of *Fourier series,* which occurs in many applications, especially in the study of wave motion (light, sound, economic cycles, clocks, oceans, etc.). They say that different frequency waves are "independent".

26. Show that $\cos x + \sin x = C\cos(x+\beta)$ for suitable constants C and β and use this to evaluate the following integrals.

(a) $\displaystyle\int \frac{\mathrm{d}x}{\cos x + \sin x}$

(b) $\displaystyle\int \frac{\mathrm{d}x}{(\cos x + \sin x)^2}$

(c) $\displaystyle\int \frac{\mathrm{d}x}{A\cos x + B\sin x}$

where A and B are any constants.

27. Compute the integrals

$$\int_0^{\pi/2} \sin^2 kx \sin^2 lx \, \mathrm{d}x,$$

where k and l are positive integers.

28. Show that for any integers k, l, m

$$\int_0^{\pi} \sin kx \sin lx \sin mx \, \mathrm{d}x = 0$$

if and only if $k + l + m$ is even.

29. (***i***) Prove the following version of the CHAIN RULE: If $f : I \to \mathbb{C}$ is a differentiable complex valued function, and $g : J \to I$ is a differentiable real valued function, then $h = f \circ g : J \to \mathbb{C}$ is a differentiable function, and one has

$$h'(x) = f'(g(x))g'(x).$$

(***ii***) Let $n \geq 0$ be a nonnegative integer. Prove that if $f : I \to \mathbb{C}$ is a differentiable function, then $g(x) = f(x)^n$ is also differentiable, and one has

$$g'(x) = nf(x)^{n-1}f'(x).$$

Note that the chain rule from part (a) does ***not*** apply! *Why?*

Answers and Hints

(2) (a) $\arg(1 + i\tan\theta) = \theta + 2k\pi$, with k any integer.

(b) $zw = 1 - \tan\theta\tan\phi + i(\tan\theta + \tan\phi)$

(c) $\arg(zw) = \arg z + \arg w = \theta + \phi$ (+ a multiple of 2π.)

(d) $\tan(\arg zw) = \tan(\theta + \phi)$ on one hand, and $\tan(\arg zw) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$ on the other hand. The conclusion is that

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

(3) $\cos 4\theta =$ real part of $(\cos\theta + i\sin\theta)^4$. Expand, using Pascal's triangle to get

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$.

(6) To prove or disprove the statements set $z = a + bi$, $w = c + di$ and substitute in the equation. Then compare left and right hand sides.

(a) $\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(z + w)$ TRUE, *because:*
$\mathfrak{Re}(z + w) = \mathfrak{Re}(a + bi + c + di) = \mathfrak{Re}[(a + c) + (b + d)i] = a + c$ and
$\mathfrak{Re}(z) + \mathfrak{Re}(w) = \mathfrak{Re}(a + bi) + \mathfrak{Re}(c + di) = a + c$.

The other proofs go along the same lines.

(b) $\overline{z + w} = \bar{z} + \bar{w}$ TRUE. *Proof:* if $z = a + bi$ and $w = c + di$ with a, b, c, d real numbers, then

$$\mathfrak{Re}(z) = a, \quad \mathfrak{Re}(w) = c \implies \mathfrak{Re}(z) + \mathfrak{Re}(w) = a + c$$
$$z + w = a + c + (b + d)i \implies \mathfrak{Re}(z + w) = a + c.$$

So you see that $\mathfrak{Re}(z) + \mathfrak{Re}(w)$ and $\mathfrak{Re}(z + w)$ are equal.

(c) $\mathfrak{Im}(z) + \mathfrak{Im}(w) = \mathfrak{Im}(z + w)$ TRUE. *Proof:* if $z = a + bi$ and $w = c + di$ with a, b, c, d real numbers, then

$$\mathfrak{Im}(z) = b, \quad \mathfrak{Im}(w) = d \implies \mathfrak{Im}(z) + \mathfrak{Im}(w) = b + d$$
$$z + w = a + c + (b + d)i \implies \mathfrak{Im}(z + w) = b + d.$$

So you see that $\mathfrak{Im}(z) + \mathfrak{Im}(w)$ and $\mathfrak{Im}(z + w)$ are equal.

(d) $\overline{zw} = (\bar{z})(\bar{w})$ TRUE

(e) $\mathfrak{Re}(z)\mathfrak{Re}(w) = \mathfrak{Re}(zw)$ FALSE. *Counterexample:* Let $z = i$ and $w = i$. Then $\mathfrak{Re}(z)\mathfrak{Re}(w) = 0 \cdot 0 = 0$, but $\mathfrak{Re}(zw) = \mathfrak{Re}(i \cdot i) = \mathfrak{Re}(-1) = -1$.

(f) $\overline{z/w} = (\bar{z})/(\bar{w})$ TRUE

(g) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$ FALSE (almost true though, only off by a minus sign)

(h) $\mathfrak{Re}(iz) = i\mathfrak{Re}(z)$ FALSE. The left hand side is a real number, the right hand side is an imaginary number: they can never be equal (except when $z = 0$.)

(i) $\mathfrak{Re}(iz) = \mathfrak{Im}(z)$ same as (g), sorry.

(j) $\mathfrak{Re}(iz) = i\mathfrak{Im}(z)$ FALSE

(k) $\mathfrak{Im}(iz) = \mathfrak{Re}(z)$ TRUE

(l) $\mathfrak{Re}(\bar{z}) = \mathfrak{Re}(z)$ TRUE

(7) The number is either $\frac{1}{5}\sqrt{5}+\frac{2}{5}i\sqrt{5}$ or $-\frac{1}{5}\sqrt{5}-\frac{2}{5}i\sqrt{5}$.

(8) 't is $\frac{1}{3}\sqrt{3}+i$.

(10) $e^{(\ln 2)(1+i)} = e^{\ln 2 + i\ln 2} = e^{\ln 2}(\cos\ln 2 + i\sin\ln 2)$ so the real part is $2\cos\ln 2$ and the imaginary part is $2\sin\ln 2$.

(11) e^z can be negative, or any other complex number except zero.

If $z = x+iy$ then $e^z = e^x(\cos y + i\sin y)$, so the absolute value and argument of e^z are $|z| = e^x$ and $\arg e^z = y$. Therefore the argument can be anything, and the absolute value can be any *positive* real number, but not 0.

(12) $\dfrac{1}{e^{it}} = \dfrac{1}{\cos t + i\sin t} = \dfrac{1}{\cos t + i\sin t}\dfrac{\cos t - i\sin t}{\cos t - i\sin t} = \dfrac{\cos t - i\sin t}{\cos^2 t + \sin^2 t} = \cos t - i\sin t = e^{-it}$.

(15) $Ae^{i\beta t} + Be^{-i\beta t} = A(\cos\beta t + i\sin\beta t) + B(\cos\beta t - i\sin\beta t) = (A+B)\cos\beta t + i(A-B)\sin\beta t$.

So $Ae^{i\beta t} + Be^{-i\beta t} = 2\cos\beta t + 3\sin\beta t$ holds if $A+B = 2, i(A-B) = 3$. Solving these two equations for A and B we get $A = 1-\frac{3}{2}i$, $B = 1+\frac{3}{2}i$.

(21) (a) $z^2+6z+10 = (z+3)^2+1 = 0$ has solutions $z = -3\pm i$.

(b) $z^3+8 = 0 \iff z^3 = -8$. Since $-8 = 8e^{\pi i + 2k\pi}$ we find that $z = 8^{1/3}e^{\frac{\pi}{3}i+\frac{2}{3}k\pi i}$ (k any integer). Setting $k = 0,1,2$ gives you all solutions, namely

$$\begin{aligned} k=0 &: \quad z = 2e^{\frac{\pi}{3}i} = 1+i\sqrt{3} \\ k=1 &: \quad z = 2e^{\frac{\pi}{3}i+2\pi i/3} = -2 \\ k=2 &: \quad z = 2e^{\frac{\pi}{3}i+4\pi i/3} = 1-i\sqrt{3} \end{aligned}$$

(c) $z^3-125 = 0$: $z_0 = 5$, $z_1 = -\frac{5}{2}+\frac{5}{2}i\sqrt{3}$, $z_2 = -\frac{5}{2}-\frac{5}{2}i\sqrt{3}$

(d) $2z^2+4z+4 = 0$: $z = -1\pm i$.

(e) $z^4+2z^2-3 = 0$: $z^2 = 1$ or $z^2 = -3$, so the ***four*** solutions are $\pm 1, \pm i\sqrt{3}$.

(f) $3z^6 = z^3+2$: $z^3 = 1$ or $z^3 = -\frac{2}{3}$. The ***six*** solutions are therefore

$$\begin{aligned} -\tfrac{1}{2}\pm\tfrac{i}{2}\sqrt{3}, 1 \quad &(\text{from } z^3 = 1) \\ -\sqrt[3]{\tfrac{2}{3}}, \sqrt[3]{\tfrac{2}{3}}\left(\tfrac{1}{2}\pm\tfrac{i}{2}\sqrt{3}\right), \quad &(\text{from } z^3 = -\tfrac{2}{3}) \end{aligned}$$

(g) $z^5-32 = 0$: The ***five*** solutions are

$$2, \quad 2\cos\tfrac{2}{5}\pi \pm 2i\sin\tfrac{2}{5}\pi, \quad 2\cos\tfrac{4}{5}\pi \pm 2i\sin\tfrac{4}{5}\pi.$$

Note that $2\cos\frac{6}{5}\pi + 2i\sin\frac{6}{5}\pi = 2\cos\frac{4}{5}\pi - 2i\sin\frac{4}{5}\pi$, and likewise, $2\cos\frac{8}{5}\pi + 2i\sin\frac{8}{5}\pi = 2\cos\frac{2}{5}\pi - 2i\sin\frac{2}{5}\pi$. (Make a drawing of these numbers to see why).

(h) $z^5-16z = 0$: Clearly $z = 0$ is a solution. Factor out z to find the equation $z^4-16 = 0$ whose solutions are ± 2, $\pm 2i$. So the ***five*** solutions are 0, ± 2, and $\pm 2i$

(22) $f'(x) = \frac{-1}{(x+i)^2}$. In this computation you use the quotient rule, which is valid for complex valued functions.

$g'(x) = \frac{1}{x} + \frac{i}{1+x^2}$

$h'(x) = 2ixe^{ix^2}$. Here we are allowed to use the Chain Rule because $h(x)$ is of the form $h_1(h_2(x))$, where $h_1(y) = e^{iy}$ is a complex valued function of a real variable, and $h_2(x) = x^2$ is a real valued function of a real variable (a "221 function").

(23) (a) Use the hint:

$$\begin{aligned} \int (\cos 2x)^4\, dx &= \int \left(\frac{e^{2ix}+e^{-2ix}}{2}\right)^4 dx \\ &= \tfrac{1}{16}\int \left(e^{2ix}+e^{-2ix}\right)^4 dx \end{aligned}$$

The fourth line of Pascal's triangle says $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Apply this with $a = e^{2ix}$, $b = e^{-2ix}$ and you get

$$\begin{aligned}\int \big(\cos 2x\big)^4 \, dx &= \frac{1}{16}\int \big\{e^{8ix} + 4e^{4ix} + 6 + 4e^{-4ix} + e^{-8ix}\big\}\, dx \\ &= \frac{1}{16}\big\{\frac{1}{8i}e^{8ix} + \frac{4}{4i}e^{4ix} + 6x + \frac{4}{-4i}e^{-4ix} + \frac{1}{-8i}e^{-8ix}\big\} + C.\end{aligned}$$

We could leave this as the answer since we're done with the integral. However, we are asked to simplify our answer, and since we know ahead of time that the answer is a real function we should rewrite this as a real function. There are several ways of doing this, one of which is to carefully match complex exponential terms with their complex conjugates (e.g. e^{8ix} with e^{-8ix}.) This gives us

$$\int \big(\cos 2x\big)^4 \, dx = \frac{1}{16}\big\{\frac{e^{8ix} - e^{-8ix}}{8i} + \frac{e^{4ix} - e^{-4ix}}{i} + 6x\big\} + C.$$

Finally, we use the formula $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ to remove the complex exponentials. We end up with the answer

$$\int \big(\cos 2x\big)^4 \, dx = \frac{1}{16}\big\{\tfrac14 \sin 8x + 2\sin 4x + 6x\big\} + C = \tfrac{1}{64}\sin 8x + \tfrac18 \sin 4x + \tfrac38 x + C.$$

(b) Use $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$:

$$\begin{aligned}\int e^{-2x}\big(\sin ax\big)^2 \, dx &= \int e^{-2x}\big(\frac{e^{iax} - e^{-iax}}{2i}\big)^2 \, dx \\ &= \frac{1}{(2i)^2}\int e^{-2x}\big(e^{2iax} - 2 + e^{-2iax}\big)\, dx \\ &= -\tfrac14 \int \big(e^{(-2+2ia)x} - 2 + e^{(-2-2ia)x}\big)\, dx \\ &= -\tfrac14\big\{\underbrace{\frac{e^{(-2+2ia)x}}{-2+2ia}}_{A} - 2x + \underbrace{\frac{e^{(-2-2ia)x}}{-2-2ia}}_{B}\big\} + C. \qquad (\dagger)\end{aligned}$$

We are done with integrating. The answer must be a real function (being the integral of a real function), so we have to be able to write our answer in a real form. To get this real form we must expand the complex exponentials above, and do the division by $-2+2ia$ and $-2-2ia$. This is still a fair amount of work, but we can cut the amount of work in half by noting that the terms A and B are complex conjugates of each other, i.e. they are the same, except for the sign in front of i: you get B from A by changing all i's to $-i$'s. So once we have simplified A we immediately know B.

We compute A as follows

$$\begin{aligned}A &= \frac{-2-2ia}{(-2-2ia)(-2+2ia)}\big(e^{-2x+2iax}\big) \\ &= \frac{(-2-2ia)e^{-2x}(\cos 2ax + i\sin 2ax)}{(-2)^2 + (-2a)^2} \\ &= \frac{e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) + i\frac{e^{-2x}}{4+4a^2}(-2a\cos 2ax - 2\sin 2ax).\end{aligned}$$

Hence

$$B = \frac{e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) - i\frac{e^{-2x}}{4+4a^2}(-2a\cos 2ax - 2\sin 2ax).$$

and

$$A + B = \frac{2e^{-2x}}{4+4a^2}(-2\cos 2ax + 2a\sin 2ax) = \frac{e^{-2x}}{1+a^2}(-\cos 2ax + a\sin 2ax).$$

Substitute this in (†) and you get the real form of the integral

$$\int e^{-2x}\big(\sin ax\big)^2 \, dx = -\tfrac14 \frac{e^{-2x}}{1+a^2}(-\cos 2ax + a\sin 2ax) + \frac{x}{2} + C.$$

(24) ***(a)*** This one can be done with the double angle formula, but if you had forgotten that, complex exponentials work just as well:

$$\begin{aligned}\int \cos^2 x \,\mathrm{d}x &= \int \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^2\,dx\\ &= \tfrac14\int\big\{e^{2ix}+2+e^{-2ix}\big\}\,dx\\ &= \tfrac14\big\{\tfrac{1}{2i}e^{2ix}+2x+\tfrac{1}{-2i}e^{-2ix}\big\}+C\\ &= \tfrac14\Big\{\frac{e^{2ix}-e^{-2ix}}{2i}+2x\Big\}+C\\ &= \tfrac14\big\{\sin 2x+2x\big\}+C\\ &= \tfrac14\sin 2x+\frac{x}{2}+C.\end{aligned}$$

(c), (d) using complex exponentials works, but for these integrals substituting $u = \sin x$ works better, if you use $\cos^2 x = 1 - \sin^2 x$.

(e) Use $(a-b)(a+b) = a^2 - b^2$ to compute

$$\cos^2 x\sin^2 x = \frac{(e^{ix}+e^{-ix})^2}{2^2}\frac{(e^{ix}-e^{-ix})^2}{(2i)^2} = \tfrac{1}{-16}(e^{2ix}+e^{-2ix})^2 = \tfrac{1}{-16}(e^{4ix}+2+e^{-4ix})$$

First variation: The integral is

$$\int\cos^2 x\sin^2 x\,dx = \tfrac{1}{-16}\big(\tfrac{1}{4i}e^{4ix}+2x+\tfrac{1}{-4i}e^{-4ix}\big)+C = \tfrac{1}{-32}\sin 4x-\tfrac18 x+C.$$

Second variation: Get rid of the complex exponentials before integrating:

$$\tfrac{1}{-16}(e^{4ix}+2+e^{-4ix}) = \tfrac{1}{-16}(2\cos 4x+2) = -\tfrac18(\cos 4x+1),$$

If you integrate this you get the same answer as above.

(j) and (l): Substituting complex exponentials will get you the answer, but for these two integrals you're much better off substituting $u = \cos x$ (and keep in mind that $\sin^2 x = 1 - \cos^2 x$.)

(k) See (e) above.

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