NODAL PROPERTIES OF SOLUTIONS OF PARABOLIC EQUATIONS

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1. Introduction. In this note we review the known facts about the zero set of a solution of a scalar parabolic equation

\[(1) \quad u_t = a(x,t) u_{xx} + b(x,t) u_x + c(x,t) u, \quad x_0 < x < x_1, 0 < t < T.\]

In particular, we discuss some applications to spectral theory, the dynamics of nonlinear diffusion equations, and the geometric heat equation for plane curves.

2. The zero number. Let \( u \) be a classical solution of (1) and assume \( u \) is continuous on the rectangle \([x_0,x_1] \times [0,T]\). Moreover, assume that

\[ u(x_i,t) \neq 0 \quad \text{for} \quad i = 0,1 \quad \text{and} \quad 0 \leq t \leq T. \]

Then, for each \( t \in [0,T] \) we define the set \( Z(t) = \{ x \in [x_0,x_1] \mid u(t,x) = 0 \} \), and we let \( z(t) \) denote the number of elements of \( Z(t) \). The set \( Z(t) \) is a compact subset of the open interval \((x_0,x_1)\).

Finally, we always assume the following about the coefficients \( a, b \) and \( c \):

\[(2) \quad a, a_x, a_{xx}, a_t, b_x, b_t \text{ and } c \text{ are continuous on } [x_0,x_1] \times [0,T]. \]

Moreover, \( a(x,t) \) is strictly positive.

In this situation we have the following:

**Theorem A.** For any \( 0 < t \leq T \), \( z(t) \) is finite. If, for some \( 0 < t_0 < T \), the function \( u(t_0) \) has a double zero, then for all

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\( t_1 < t_0 \leq t_2 \) we have \( z(t_1) > z(t_2) \). Here a double zero is a point where both \( u \) and \( u_x \) vanish.

This theorem shows that the number of zeros, \( z(t) \), does not increase with time. The theorem is a refinement of a result of Nickel, Matano and Henry (see [7, 6, 5]). If the coefficients and the solution are real analytic, then Theorem A was proven in [3]. The general case was proven in [2].

The idea of the proof in the analytic case is to study the Taylor series of a solution \( u(t, x) \) near its multiple zeros. If \((\hat{t}, \hat{x})\) is such a zero, then repeated differentiation of the equation (1) shows that, up to rescaling and higher order terms, one has

\[
(3) \quad u(\hat{t} + \tau, \hat{x} + \delta) = \frac{\delta^m}{m!} + \frac{\tau}{1!} \frac{\delta^{m-2}}{(m-2)!} + \frac{\tau^2}{2!} \frac{\delta^{m-4}}{(m-4)!} + \ldots
\]

with \( m \geq 2 \). Using the Newton polygon method, one then finds that the zero set of \( u(x, t) \) near \((\hat{t}, \hat{x})\) consists of a finite number of curves. Furthermore, if \( m \) is even, all these curves lie in the region \( t \leq \hat{t} \). If \( m \) is odd, there is one additional curve that intersects the line \( t = \hat{t} \) transversally (see Figure 1). In either case, the number of zeros of \( u(t, \cdot) \) drops as \( t \) increases beyond \( \hat{t} \).

It should be noted that the polynomials given in [3] are special solutions of the heat equation \( u_\tau = u_{\delta x} \) and that it can be instructive to study their graphs (see Figure 2).

The boundary conditions \( u(x_0, t) \neq 0 \) are not the only ones under which Theorem A holds. More general conditions were discussed in [2], and one we would like to mention here is the periodic case.

If the functions \( u, a, b \) and \( c \) are periodic in \( x \) with period 1 (so that they are defined on \( \mathbb{R} \times [0, T] \)) and satisfy (2) on \( \mathbb{R} \times [0, T] \) instead of \([x_0, x_1] \times [0, T] \), then Theorem A remains valid if one defines \( z(t) \) to be the number of zeros of \( u(t, \cdot) \) in the interval \([0, 1]\).

3. **Time-dependent Sturm Liouville theory.** Let \( c(x, t) \) be a continuous function on \( \mathbb{R} \times [0, T] \) satisfying \( c(x + 1, t) \equiv c(x, t) \). Then we define a linear operator \( L \) on \( C(\mathbb{R}/\mathbb{Z}) \) by the following recipe. Given
FIGURE 1. The zeroset near a multiple zero.

FIGURE 2. Some special solutions of the heat equation.
$f \in C(\mathbb{R}/Z)$ one computes $Lf$ by solving the initial value problem

$$u_t = u_{xx} + c(x,t)u, \quad x \in \mathbb{R}/Z, \quad t \in [0,T]$$

(4) \hspace{1cm} u(x,0) = f(x)

and defining $Lf(x) = u(x,T)$.

Standard results on the smoothing property of parabolic equations imply that $L$ is a bounded compact operator on $E = C(\mathbb{R}/Z)$. Thus, its spectrum consists of, at most, a countable number of eigenvalues, clustering at $\lambda = 0$. We denote these eigenvalues by $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots$ and order them so that $|\lambda_j| \geq |\lambda_{j+1}|$. Each eigenvalue is assumed to occur as often as its algebraic multiplicity.

If $c(x,t) = c(x)$ does not depend on $t$, then we may write $L = \exp(A)$ where $-A$ is the Hill’s operator $-A = -(d/dx)^2 - c(x)$. In this case it is known that the eigenvalues $\lambda_j$ come in pairs, i.e., $\lambda_{2n} > \lambda_{2n+1}$ for all $n \geq 0$. Also, the eigenfunctions belonging to $\lambda_{2n-1}$ and $\lambda_{2n}$ have exactly $2n$ zeros in one period interval $0 \leq x < 1$.

Using Theorem A one can show that this also holds in the general case where $c$ does depend on time. More precisely, if $L$ is defined as above, then we have

(5) \hspace{1cm} \lambda_0 > |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq |\lambda_4| > \cdots.

In particular, for any $n \geq 1$, $\{\lambda_{2n-1}, \lambda_{2n}\}$ is a spectral set for the operator $L$, so that its corresponding spectral subspace $F_n \subset E$ is well defined. This space is two dimensional and any real function $f \in F_n$ has exactly $2n$ zeros, all of which are simple.

The proof of these statements is contained in [1,3]. The key ingredient is the following observation: for any $f \in E$, $Lf$ has only a finite number of zeros, and $z(Lf) \leq z(f)$. If $Lf$ has a multiple zero, then $z(Lf) < z(f)$. This follows from Theorem A and the definition of $L$.

It immediately implies that, if $f$ is a real eigenfunction, its zeros are all simple (since $z(f) = z(Lf)$). A lengthier argument along the same lines leads to the statements we just made.

4. Rotating waves. We consider the initial value problem

$$u_t = f(u, u_x, u_{xx}), \quad x \in \mathbb{R}/Z, t > 0$$

(6) \hspace{1cm} u(x,0) = u_0(x)$$
in which \( f(u, p, s) \) is a \( C^\infty \) function of its arguments and \( f_4(u, p, s) > 0 \).

In [3] the dynamics of the semiflow generated by such an equation was studied in the semilinear case (i.e., \( f(u, p, s) = s + g(u, p) \) for some other function \( g \)).

Using Theorem A we can prove the following: Let \( u(t, x) \) be a periodic solution of (6), i.e., \( u(t+T, x) \equiv u(t, x+1) \equiv u(t, x) \) and suppose \( u \) is so smooth that \( u_t \) and \( u_{xx} \) are H"older continuous. By parabolic regularity theory the solution \( u \) is then actually \( C^\infty \). We now have:

\[
u \text{ is either constant, or a rotating wave, i.e.,}
\]

\[
\text{of the form } U(x - ct) \text{ for some } c \in \mathbb{R}.
\]

To prove this we observe that any linear combination \( w \) of \( u_t \) and \( u_x \) is a solution of

\[
\begin{align*}
  w_t &= a \cdot w_{xx} + b \cdot w_x + c \cdot w \\
  w(x+1, t) &= w(x, t+T) \equiv w(x, t)
\end{align*}
\]

(7)

where \( a = f_4(u, u_x, u_{xx}) \), \( b = f_p(u, u_x, u_{xx}) \) and \( c = f_u(u, u_x, u_{xx}) \).

So if \( w \not\equiv 0 \), then for any time \( t \), \( w(\cdot, t) \) has only a finite number of zeros, \( z(t) \).

Furthermore, \( z(t) \) is nonincreasing, and by periodicity \( z(t+T) = z(t) \). Hence, \( z(t) \) must be constant, and Theorem A implies that \( w(\cdot, t) \) never has a multiple zero.

Now choose a point where \( u(x, t) \) attains its maximal value, say \((x_0, t_0)\). Then both \( u_x \) and \( u_t \) vanish at \((x_0, t_0)\) and there must be a linear combination \( w = au_x + \beta u_t \) such that \( w_x \) also vanishes at this point. The foregoing considerations show that \( w \equiv 0 \), and we are left with two cases. If \( \beta = 0 \), then \( w = au_x \equiv 0 \), so that \( u \) is constant. Otherwise, we have \( u_t + cu_x = 0 \) with \( c = \alpha/\beta \) so that \( u \) can be written as \( u(x - ct) \).

In [3] many other results were derived; in particular, the existence of connecting orbits between different rotating waves was studied.

5. The geometric heat equation. Let \( X \) be a regular curve in the plane, i.e., a \( C^1 \) mapping from \( \mathbb{R}/Z \) into \( \mathbb{R}^2 \) whose derivative never vanishes. The curve may have self-intersections.
We shall use the letter \( u \) to denote the parameter in \( \mathbb{R}/Z \) on the curve (i.e., \( X = X(u), u \in \mathbb{R}/Z \)).

If the curve is \( C^2 \), then its curvature \( k \) is well defined. The geometric heat equation is the following

\[
\frac{\partial X}{\partial t} = kN \quad \text{or} \quad \frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2}
\]

where \( N \) is the unit normal to the curve, and \( s \) denotes arclength along the curve. The second form of the equation is slightly misleading since \( \partial / \partial t \) stands for a derivative w.r.t. \( t \) with constant \( u \in \mathbb{R}/Z \), and not constant \( s \). A more precise version is

\[
X_t = |X_u|^{-1}(|X_u|^{-1}X_u)_u, \quad X(u + 1, t) \equiv X(u, t) \\
u \in \mathbb{R}/Z, \ t > 0.
\]

This is a degenerate system of parabolic PDEs. Local solvability in time was shown in [4] for \( C^\infty \) initial data.

It is known that, if \( X(u, t) \) \((0 \leq t < T)\) is a solution of (9) whose initial value has no self-intersections, then for all \( 0 < t < T \), the curve \( X(\cdot, t) \) also has no self-intersections (see [4]).

Using Theorem A we can say a little more.

Let \( X(u, t) \) be a solution of (9). Choosing rectangular coordinates \( x, y \) in the plane any small enough portion of the family of curves \( X(u, t) \) can be represented as the graph of a function \( y = w(x, t) \). A lengthy computation shows that (9) is, locally at least, equivalent to the following equation for \( w \).

\[
w_t = \frac{w_{xx}}{1 + (w_x)^2} \overset{\text{def}}{=} F(w_x, w_{xx}).
\]

Since this is a quasilinear parabolic equation, the curves \( X(\cdot, t) \) are, for each \( t \), real analytic.

If we have two solutions of (9), say \( X_1 \) and \( X_2 \), then for any \( t > 0 \), they either coincide or they have only a finite number of intersections, say \( i(t) \).

Near a point of intersection both curves can be represented by two solutions \( w^1 \) and \( w^2 \) of (10) (if one chooses the \( y \)-axis in the right direction).
The difference \( v = w^1 - w^2 \) satisfies a linear equation of the form

\[
v_t = a(x,t)v_{xx} + b(x,t)v_x
\]

(just subtract equation (10) for \( w^1 \) and \( w^2 \), and apply the mean value theorem to \( F \)).

By Theorem A, the number of zeros of \( v \) cannot increase and in fact decreases if \( v(t, \cdot) \) has a multiple zero. Since zeros of \( v(t, \cdot) \) correspond to intersections of \( X_1 \) and \( X_2 \), we arrive at the following conclusion.

At any time \( t > 0 \) for which the curves \( X_1 \) and \( X_2 \) are defined, their number of intersections, \( i(t) \), is finite.

If for some \( t_0 > 0 \), \( X_1 \) and \( X_2 \) have a nontransversal intersection, then \( i(t) \) drops as \( t \) increases \( t_0 \).

A similar argument shows that the number of self-intersections of a solution \( X(t,u) \) of (9) cannot increase with time.

To conclude this discussion we note that the curvature \( k \) as a function of normalized arclength \( s \) satisfies

\[
k_t = k_{ss} + (\beta k)_s = k_{ss} + \beta k_s + \beta_s k
\]

where \( \beta(s,t) = \int_0^s k(s', t)^2 ds' - s \int_0^1 k(s', t)^2 ds' [1] \).

The normalized arclength is defined to be ordinary arclength divided by the total length of the curve. Thus \( k \) and \( \beta \) are periodic functions of \( s \), with period 1.

If we apply Theorem A to (11), then we find:

for any \( t > 0 \) the curve \( X(\cdot, t) \) has a finite number of flexpoints.

This number does not increase with time.

(Recall that a flexpoint is a point on the curve where \( k \) vanishes.)

Differentiating (11) with respect to \( s \), and using \( \beta_{ss} = 2k \cdot k_s \), we see that \( k_s \) also satisfies an equation of the form (1) so that Theorem A can again be applied.

For any \( t > 0 \) the curve \( X(\cdot, t) \) has a finite number of vertices.

This number does not increase with time.

(A vertex of a plane curve is a point where the curvature reaches a local maximum or minimum [8, Vol. 2].)
REFERENCES


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