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Shrinking Doughnuts.

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Introduction.

Let M^n be a smooth compact oriented manifold, and let $X : M \times [0,T) \to \mathbf{R}^{n+1}$ be a smooth family of immersions of M in $n+1$ dimensional Euclidean space. The orientation of M allows one to define a unique smooth unit normal vector field $\nu_X : M \times [0,T) \to \mathbf{R}^{n+1}$. Given this choice of ν_X, we can define the principal curvatures, $\kappa_1, \ldots, \kappa_n$, of the immersion $X(\cdot, t)$ and the mean curvature $H_X = (\kappa_1 + \ldots + \kappa_n)/n$ in the usual way. By definition, the family of immersions $X(\cdot, t)$ "moves by its mean curvature" if the normal velocity satisfies

$$\langle X_t(p,t), \nu_X(p,t)\rangle = nH_X(p,t) \tag{1}$$

for all $(p,t) \in M \times [0,T)$. Here $\langle x, y\rangle = x_0y_0 + \cdots + x_ny_n$ denotes the Euclidean inner product on $\mathbf{R}^{n+1}$.

The object of this note is to study some of the similarity solutions of (1), and in particular to show that there exists an embedding of the 2-torus, $X_0 : \mathbf{T}^2 \to \mathbf{R}^3$ for which the corresponding solution to (1) is simply given by $X(p,t) = \sqrt{2(1-t)}X_0(p)$, i.e., for which the torus will shrink to the origin by dilations, and for which it will become singular at $t = 1$. More precisely, we'll prove the following.

Theorem. *For $n \geq 2$ there exist embeddings $X_n : S^1 \times S^{n-1} \to \mathbf{R}^{n+1}$ for which $X_n(p,t) = \sqrt{2(1-t)} \cdot X_n(p)$ is a solution of the flow by mean curvature equation.*

We will show how the existence of such solutions can be exploited to study the singularities of general solutions of (1).

Similarity solutions of (1) were studied both by Abresch and Langer [AL], and by Epstein and Weinstein [EW] in the case of curves in the plane. The higher dimensional situation was studied by Huisken in [Hu2],

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in which paper he also obtained some detailed results on the formation of singularities in the flow by mean curvature problem. One of Huisken's results in [Hu2] states that the only immersed hypersurface with positive mean curvature which gives rise to a similarity solution is the standard n sphere. Quoting numerical evidence for the existence of toroidal similarity solutions, obtained by Matt Grayson, Huisken then points out that his hypothesis concerning the positivity of the mean curvature appears to be necessary. Our existence result confirms this.

Before we discuss the toroidal similarity solutions, we give a short overview of some facts about the flow by mean curvature problem which have been established recently. Our discussion is necessarily incomplete: in particular we won't mention the recent interesting theories of "weak solutions" of Evans and Spruck [ES], and Chen, Giga and Goto [CGG]. The section on "the one dimensional case", reflects the contents of my talk at Gregynog.

ACKNOWLEDGEMENT. The preprint [Hu2] has inspired some of what follows, and it is a pleasure to thank Bernd Kawohl for sending me a copy of Huisken's work.

The initial value problem.

One can regard the equation (1) as a degenerate parabolic partial differential equation for X. The degeneracy is caused by the fact that (1) is invariant under reparametrizations; if $\varphi : M \times [0, T) \to M \times [0, T)$ is a smooth one parameter family of diffeomorphisms of M, then $X(p, t)$ will satisfy (1) if and only if $X^\varphi(p, t) = X(\varphi(p, t), t)$ does so. This shows that the solution to (1) is not unique, and in fact that it can at most be unique up to reparametrizations of the form $X^\varphi(p, t) = X(\varphi(p, t), t)$.

The degeneracy can be removed by restricting oneself to a specific class of immersions X. One possible choice is the following. Let $Y : M \to \mathbf{R}^{n+1}$ be a given immersion. Then by the tubular neighborhood theorem there is a small $\epsilon > 0$ such that the map $\sigma : M \times (-\epsilon, \epsilon) \to \mathbf{R}^{n+1}$ given by $\sigma(p, u) = Y(p) + u\nu_Y(p)$ is a local diffeomorphism. Any immersion $X_0 : M \to \mathbf{R}^{n+1}$ which is C^1 close to Y can then be represented as the graph of a C^1 function u_0 on M, i.e. as $X_0(p) = \sigma(p, u_0(p))$. If one has a family of such immersions, then they will be represented by a function $u : M \times [0, T) \to \mathbf{R}$, at least as long as they stay close to the "reference immersion" Y. If one computes the mean curvature and unit normal corresponding to an immersion of the form $X_0(p) = \sigma(p, u_0(p))$, then one finds that, in local coordinates $x_1, \ldots, x_n$ on M, (1) is equivalent to

$$\frac{\partial u}{\partial t} = g^{ij}(x, u, \partial u)\frac{\partial^2 u}{\partial x_i \partial x_j} + A(x, u, \partial u), \tag{1'}$$

where $[g^{ij}]_{1\le i,j\le n}$ is the inverse of $[g_{ij}]_{1\le i,j\le n}$, and the g_{ij} are the components of the metric which the immersion $p \mapsto X(p, t) = \sigma(p, u(p, t))$ induces on M. The lower order term $A(x, u, \partial u)$ depends on the reference immersion Y in a fairly complicated way; the term vanishes in the special case when $Y(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$ is a hyperplane (but this example can only be used to study the evolution of graphs in $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ – if M is compact then $Y : M \to \mathbf{R}^{n+1}$ can never be represented as such a graph.) Using the extant theory of quasilinear parabolic partial differential equations, one can now prove a "short time" existence result for $(1')$, assuming that the initial immersion $X_0 : M \to \mathbf{R}^{n+1}$ has bounded principal curvatures, say. Huisken [Hu1] showed that the solution will exist as long as its principal curvatures remain bounded. On the other hand, one can show that the maximal classical solution which one obtains in this way must become singular within finite time. Naturally, one wants to know *how* this solution can become singular.

Assuming that M is a sphere, that $n \geq 2$, and that the initial immersion is a strictly convex embedding, Huisken [Hu1] showed that the solution remains convex, shrinks to a point, and assumes the shape of a sphere with radius approximately equal to $\sqrt{c(T-t)}$, as $t \uparrow T$. Roughly at the same time Gage and Hamilton [GH] gave a proof of the same fact in the case $n = 1$ (the so called "curve shortening" problem); they showed that a convex curve will shrink to a point and become asymptotically circular as it shrinks. In fact, Gage had shown earlier that the isoperimetric ratio of the curve is decreasing. Gage and Hamilton also observed that, if one starts with a simple closed curve, then the corresponding solution to (1) never develops a self intersection before it becomes singular. This result was then strengthened by Grayson: he showed that, if one starts with any simple closed curve,then the solution actually becomes convex before it can become singular, so that by Gage and Hamilton's result it will eventually shrink to a "round point".

Figure 1.

It was later found that one cannot extend Grayson's theorem to the higher dimensional situation, or, to put it differently, one cannot remove the convexity hypothesis from Huisken's theorem. The following counter example shows why. Assume that M is indeed a sphere, and suppose that the initial immersion X_0 has the shape suggested in figure 1: two spheres connected by a very thin tube. Intuitively it is clear that the two spherical parts of the surface will evolve slowly (their curvature is not very large), while the thin tube will tend to collapse. Indeed, one of its curvatures (parallel to its axis) is rather small, and the other is very large, and directed "inwards".

It turns out to be nontrivial to prove rigorously that these effects will really cause the "neck" to be pinched off. As far as I know, Matt Grayson [Gr2] was the first to provide such a proof. At this conference Bernd Kawohl has shown us another proof, using the classical maximum principle; in addition he showed us (some of) the output of a numerical simulation done by Huisken and Alt, which ought to convince anyone that the thin "neck" in figure 1 will really break. In [Hu2] Huisken has given the best results in this direction so far, of which I am aware. Assuming that the maximal curvature does not blow up faster than $c(T-t)^{-1/2}$, he shows that the asymptotic shape of the solution near a blow up point is given by a self similar solution. In the case of a two dimensional surface of rotation in $\mathbf{R}^3$, whose shape resembles the surface of figure 1, Huisken shows that the curvature will actually not blow up faster than $c(T-t)^{-1/2}$; moreover, the shape of the "neck" near a blow up point will converge to a cylinder, after it is magnified by a factor $(T-t)^{-1/2}$.

The case of curves in the plane.

It was observed by Gage and Hamilton [GH] that the evolution of plane curves under the (mean) curvature flow can be simplified if one restricts ones attention to convex curves, i.e. to immersed curves without inflection points. On such a curve the angle θ which the tangent makes with a fixed direction, such as the x- axis, is a good coordinate, and the curve is determined up to a translation, if one specifies the curvature k as a function of the angle θ.

Given the curvature $k(\theta)$ one finds the arc length s as a function of θ by integrating $ds = d\theta/k$; a parametrization of the curve is then given by

$$x(s) = x_0 + \int_0^\theta \frac{\cos\theta}{k(\theta)}\,d\theta; \qquad y(s) = y_0 + \int_0^\theta \frac{\sin\theta}{k(\theta)}\,d\theta,$$

where (x_0, y_0) are the coordinates of the point with $\theta = 0$. Conversely, any positive continuous $2\nu\pi$ periodic function function $k(\theta)$ which satisfies

$$\int_0^{2\nu\pi} \frac{e^{i\theta}}{k(\theta)}\,d\theta = 0$$

defines a closed curve of winding number ν (number of times the tangent runs through the unit circle, as you go around the curve once).

The curve shortening problem turns out to be equivalent to the following parabolic partial differential equation for $k(\theta, t)$

$$k_t = k^2 k_{\theta\theta} + k^3, \tag{2}$$

where the variable θ belongs to $\mathbf{R}/2\nu\pi\mathbf{Z}$, if ν is the winding number of the curve ; alternatively, $k(\theta, t)$ should satisfy periodic boundary conditions $k(\theta + 2\nu\pi, t) \equiv k(\theta, t)$.

In a different context this equation was studied by A. Friedman and B. McLeod, in [FM], which inspired some of the results obtained in [An2].

The similarity solutions which were found and studied by Abresch and Langer, and by Epstein and Weinstein correspond to the solutions of (2) which one obtains if one tries $k(\theta, t) = f(t)K(\theta)$. One finds that $f(t)$ must be $\{2(1-t)\}^{-1/2}$, while K should be a solution of

$$K''(\theta) + K(\theta) = \frac{1}{2K(\theta)}.$$

These are not the only special solutions of (2): by looking for time independent solutions of (2) one easily finds that $k(\theta, t) = A\cos(\theta - \theta_0)$ satisfies (2) for any $A > 0, \theta_0 \in \mathbf{R}$. This solution corresponds to a curve which is essentially (i.e. up to translation, rotation and dilation) the graph of $y = -\log\cos x$. Its evolution under the mean curvature flow is given by translating it with constant velocity, parallel to its asymptotes. This solution is now refered to as the "Grim Reaper."

A less obvious solution to (2) is:

$$k(\theta, t) = \sqrt{\cos 2\theta - \coth 2t} \qquad (-\infty < t < 0, \theta \in \mathbf{R}).$$

The shape of this solution may be described as follows. For $t = -\infty$ the corresponding curve consists of two Grim Reapers, with the same asymptotes, but separated by an infinite distance; as t increases from $-\infty$ to 0, the two Grim Reapers will move toward each other, and for t close to 0, the curve will become asymptotically circular, and shrink to a point. It is not clear to me whether this solution can be obtained as some sort of "similarity solution" of (2).

In the one dimensional case one might hope that Grayson's theorem could be extended to curves with self intersections. Indeed, as we have just mentioned, Abresch and Langer found that there are solutions, analogous to the circle, which shrink in a self similar way to a point. Nevertheless, it can be made quite plausible by intuitive arguments, that curves with self intersections will in general become singular without shrinking to a point: small loops have larger curvature, and hence should contract faster than the remainder of the curve. The sequence in figure 2 indicates what can

Figure 2 – A singularity in the curve shortening problem.

happen. The formation of such singularities was studied by this author in [An1, An2].

In [An2] we considered strictly convex immersions of the circle in $\mathbf{R}^2$, and used equation (2) to study how their corresponding solutions become singular. The results may summarized as follows.

Let $\kappa(t)$ be the maximal curvature of the curve at time t, and assume that the curve becomes singular at time T. *If* $\sqrt{T-t}\cdot\kappa(t)$ *remains bounded, then the curve will shrink to a point, and its asymptotic shape will be one of the self similar solutions found by Abresch and Langer.* Thus self similar blow up occurs if and only if the curvature blows up like $(T-t)^{-1/2}$.

To describe the situation in which $\kappa(t)\sqrt{T-t}$ is not bounded, we define a "normalized" curve for each $t < T$. Choose a point $P(t)$ on the curve where the curvature attains its maximum, rotate and translate the curve so that this point becomes the origin, so that its tangent becomes horizontal, and so that it is curved upwards at the origin; next magnify the curve so that its curvature at the origin becomes $+1$. We shall call the curve thus obtained $\Gamma(t)$.

It was shown in [An2] that, *if* $\sup_{t<T}\sqrt{T-t}\cdot\kappa(t) = \infty$, *then there exists a sequence* $t_n \uparrow T$ *such that the curves* $\Gamma(t_n)$ *converge to the graph of* $y = -\log\cos x$ $(|x| < \pi/2)$.

Finally, if one defines the *blow up* set of a solution $k(\theta, t)$ of (2) to be the set $\Sigma = \{\theta \in \mathbf{R}/2\nu\pi\mathbf{Z} : \lim_{t\uparrow T} k(\theta, t) = \infty\}$, then one can show that Σ consists of a finite number of intervals, each of which has length at least π (see [GH, An2]). Theorem D of [An2] says that, if $|\Sigma| < 2\pi$, then one actually has $|\Sigma| = \pi$; in this situation the curves $\Gamma(t)$ will converge to the "Grim Reaper," i.e. to the graph of $y = -\log\cos x$. (This is stronger than the previous statement which only said that some subsequence $\Gamma(t_n)$ converges.) Moreover, one obtains the following estimate for the rate with which the curvature blows up:

$$\lim_{t\uparrow T}(T-t)^{1/2+\epsilon}\kappa(t) = \begin{cases} \infty, & \text{if } \epsilon = 0; \\ 0, & \text{for all } \epsilon > 0. \end{cases}$$

The results in [An2] are definitely not the last word on "blow up" for (2). Many questions remain, such as "what is the precise blow up rate?", and "which blow up sets Σ can occur?" (guess: Σ must be the union of disjoint intervals whose lengths are multiples of π). Besides answering such questions one would also like to remove the convexity hypothesis concerning the curves, which allows one to use the convenient equation (2).

Self similar solutions.

By definition a self similar solution of (1) is a solution which can be parametrised as $X(p,t) = f(t)X_0(p)$. By inspection one finds that the only $f(t)$'s which can occur here are given by $f(t) = \sqrt{2(T-t)}$; without loss of generality we may assume that $T = 1$. Again, by substituting the *ansatz* $X(p,t) = f(t)X_0(p)$ in (1), one finds that $X(p,t) = \sqrt{2(1-t)} \cdot X_0(p)$ is a solution of (1) if and only if the immersion $X_0 : M \to \mathbf{R}^{n+1}$ satisfies

$$nH_{X_0}(p) + \frac{1}{2}\langle X_0(p), \nu_{X_0}(p)\rangle = 0 \tag{3}$$

for all $p \in M$. Except for the lower order term $\frac{1}{2}\langle X_0(p), \nu_{X_0}(p)\rangle$, this is the equation which determines minimal hypersurfaces. In fact the solutions to (3) are exactly the immersions $X_0 : M \to \mathbf{R}^{n+1}$ at which the functional

$$A(X) = \int_M e^{-|X(p)|^2/4} d\sigma_X^n(p)$$

is stationary (cf. [Hu2, Theorem 3.1]). Here $d\sigma_X^n$ denotes the n dimensional volume element which $X : M \to \mathbf{R}^{n+1}$ induces on M.

Indeed, a straightforward calculation shows that the first variation of $A(X)$ under a normal variation $X(\epsilon, p) = X(p) + \epsilon u(p)\nu_X(p)$ for any given $u \in C^\infty(M)$ is given by

$$\left.\frac{dA(X+\epsilon u\nu_X)}{d\epsilon}\right|_{\epsilon=0} = -\int_M e^{-|X(p)|^2/4}[nH_{X_0}(p) + \frac{1}{2}\langle X_0(p), \nu_{X_0}(p)\rangle]u(p)\, d\sigma_X^n(p) \tag{4}$$

In other words, the solutions to (3) are exactly the minimal hypersurfaces in $\mathbf{R}^{n+1}$ with respect to the metric $(ds)^2 = e^{-|x|^2/4n}\{(dx_0)^2 + \ldots + (dx_n)^2\}$.

We shall not consider the most general solution of (3); instead, we shall restrict our attention to hypersurfaces of revolution. This means that we assume that $M = S^{n-1} \times (a, b)$, and that the immersion X_0 has the form $X_0(\omega, s) = x(s)e_0 + r(s)\omega$, i.e. that X_0 is obtained by rotating the plane curve parametrised by $(x(s), r(s))$ around the x_0 axis. We have identified S^{n-1} with the standard unit sphere in $\{0\} \times \mathbf{R}^n \subset \mathbf{R}^{n+1}$, and e_0 denotes the first unit basis vector $(1, 0, \ldots, 0)$ in $\mathbf{R}^{n+1}$.

The functional $A(X)$ will be stationary at the immersion X corresponding to $(x(s), r(s))$ if and only if the curve $\{(x(s), r(s)) : s \in (a, b)\}$ is a geodesic in the upper half plane $\{r > 0\}$ with metric

$$(ds)^2 = r^{2(n-1)} e^{-(x^2+r^2)/4} \{(dx)^2 + (dr)^2\}. \tag{5}$$

Indeed, the volume element $d\sigma_X^n$ is given by

$$d\sigma_X^n = r(s)^{n-1} \sqrt{x'(s)^2 + r'(s)^2} \cdot ds\, d\omega^{n-1},$$

where $d\omega^{n-1}$ is the volume element on the $n-1$ sphere. Therefore one can write $A(X)$ as

$$A(X) = \mathrm{vol}(S^{n-1}) \int_a^b r(s)^{n-1} \sqrt{x'(s)^2 + r'(s)^2}\, ds$$

which up to a multiplicative constant is the length of the curve parametrized by $(x(s), r(s))$ in the metric (5). Thus if $A(X)$ is sationary at some immersion X which comes from a hypersurface of rotation, then the corresponding curve will certainly be a geodesic of (5). Conversely, if one has a geodesic, then the corresponding immersion X will be a stationary point of A under rotationally symmetric variations of X. Using (4) one shows that, if X is rotationally symmetric, then

$$\left.\frac{dA(X + \epsilon u(s,\omega)\nu_X)}{d\epsilon}\right|_{\epsilon=0} = \left.\frac{dA(X + \epsilon \bar{u}(s)\nu_X)}{d\epsilon}\right|_{\epsilon=0}$$

where $\bar{u}(s)$ is the average of $u(s, \omega)$ over S^{n-1}. In other words, if A is stationary at X under symmetric variations, then it is stationary under all variations.

At this point we can formulate the main result of this note.

Theorem. *Let $n \geq 2$. The upper half plane, equipped with the metric* (5)*, has at least one simple closed geodesic which is symmetric with respect to reflection in the r-axis.*

One should observe that the metric (5) is not complete (the r-axis has finite length). One should also note that in the case $n = 1$, which corresponds to the curve shortening problem, the metric extends to a smooth metric on the entire plane. This metric is invariant under rotations, so that its geodesic flow is integrable: all geodesics are either closed, or are densely wound on tori of constant angular momentum. In the case which we shall be considering, $n \geq 2$, the metric admits no obvious symmetry, and a simple minded numerical study of the geodesic flow (done on a Macintosh SE/30 in Turbo Pascal) indicates that the flow does not seem to have a second integral (the energy $r^{n-1}e^{-(x^2+r^2)/8}\sqrt{\dot{x}^2+\dot{r}^2}$ being the first integral). Figure 3 shows some geodesic curves for the metric (5) with $n = 2$.

To prove our theorem, we shall use a "shooting method". Given any point (x, r) in the upper halfplane, and any angle $\theta \in \mathbf{R}$ there exists a unique geodesic through (x, r) whose tangent at that point is $(\cos\theta, \sin\theta)$. If one parametrizes such geodesics by arc length, then one obtains a flow on the set of unit tangent vectors, the geodesic flow of the metric (5) on the unit tangent bundle. Using Liouville's formula ([DoC, p. 253]) one finds that this flow is given by the following system of ordinary differential equations:

$$
\begin{cases}
\dot{x} = \cos\theta \\
\dot{r} = \sin\theta \\
\dot{\theta} = \dfrac{x}{2}\sin\theta + \left(\dfrac{n-1}{r} - \dfrac{r}{2}\right)\cos\theta
\end{cases}
\tag{6}
$$

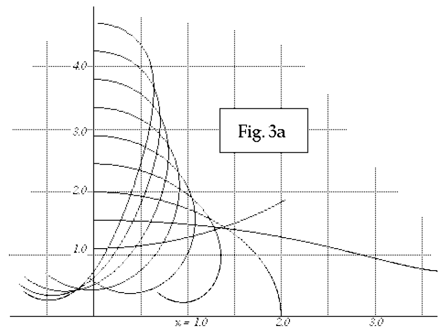
Let $(x_R, r_R, \theta_R) = \Gamma_R : [0, T(R)) \to \mathbf{R}^3$ be the maximal solution of (6) with initial value $\Gamma_R(0) = (0, R, 0)$, and let $\gamma_R(t) = (x_R(t), r_R(t))$ be the projection of Γ_R on the xr- plane. Then we intend to show that there exist $R > 0, t_R > 0$ such that $\gamma_R([0, t_R])$ is a simple curve in the first quadrant which begins and ends on the r-axis, and whose tangents on the r-axis are horizontal, i.e. perpendicular to the r-axis. Since reflection in the r-axis, $(x, r) \mapsto (-x, r)$, is an isometry for the metric (5), the curve obtained by reflecting $\gamma_R([0, t_R])$ in the r-axis is a closed geodesic whose existence is claimed by the theorem.

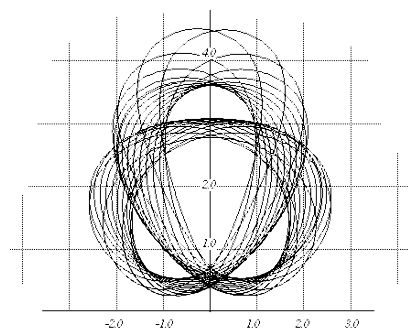
Special solutions. The metric (5) has a few simple geodesics whose existence will help us in our proof. They are the following. First, the r-axis is a geodesic: refering to the original flow by mean curvature problem (1), this geodesic corresponds to a plane through the origin. Such a plane is an equilibrium for (1), and hence may be considered as a self similar solution.

It is well known that an n sphere in $\mathbf{R}^{n+1}$ will shrink to its centre by dilations, so there should be a geodesic for (5) corresponding to a sphere; it is given by the circle $x^2 + r^2 = 2n$.

Finally, a cylinder $\mathbf{R} \times S^{n-1}$ centered at the x_0 axis in $\mathbf{R}^{n+1}$ will also shrink by dilations. This is reflected in the fact that the straight line $r = \sqrt{2(n-1)}$ in the upper half plane is a geodesic for (5).

General behaviour of Γ_R. We shall always assume that $R > \sqrt{2(n-1)}$. Under this restriction one has $\theta'(0) < 0$, so that the curve γ_R will initially bend downwards. Then, as long as $x > 0$, $\theta > -\pi/2$ and $r > \sqrt{2(n-1)}$ hold, one will have $\theta' < 0$, so that the curve will be convex (in the common, Euclidean, sense).





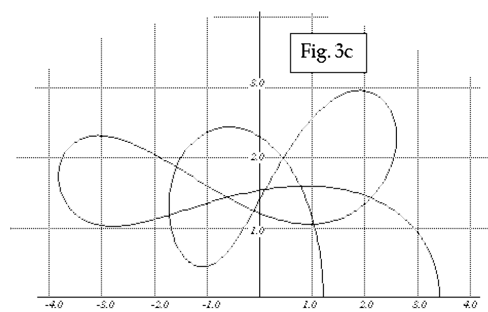


Figure 3a shows some of the geodesic segments γ_R which are used in the proof of the main theorem.

Figure 3b exhibits a geodesic segment which appears to come from a quasi periodic orbit of the system (6).

Figure 3c shows a geodesic which when rotated around the x-axis, would lead to a self intersecting immersed 2-sphere in $\mathbf{R}^3$, which generates a similarity solution of the flow by mean curvature equation. By experimenting one easily finds numerical evidence for many more of these solutions.

Let $t_1 = t_1(R) > 0$ be the first time, if any, at which either $x_R = 0$ or $\theta_R = 0$, or $\theta_R = -\pi$ occurs (if this never happens, then we put $t_1(R) = T(R)$.) On the segment $\gamma_R[(0, t_1(R))]$ one has $0 > \theta > -\pi$, so that this segment is the graph of a function $x = f_R(r)$, defined for $r_R(t_1(R)) < r < R$.

At any point where this function is stationary one has $\theta = -\pi/2$, so that $\theta' = -x/2 < 0$, which implies $f''_R < 0$. In other words, f_R can only have local maxima, and hence it can have at most one critical point, which must then be a maximum.

Asymptotics as $R \uparrow \infty$. In this section we show that, for large enough R, the geodesic γ_R will make a sharp downward bend at $(0, R)$, and then follow the r-axis closely for a while, until it intersects the r-axis, somewhere above the line $r = 1/R$.

Since R is large, we put $R = \epsilon^{-1}$, and we introduce the variables $\xi(\tau) = Rx(\epsilon\tau)$, $\rho(\tau) = R(r(\epsilon\tau) - R)$ and $\vartheta(\tau) = \theta(\epsilon\tau)$. They satisfy

$$\begin{cases} \dot{\xi} = \cos\vartheta \\ \dot{\rho} = \sin\vartheta \\ \dot{\vartheta} = -\dfrac{1}{2}\cos\vartheta + \mathcal{O}(\epsilon^2) \end{cases} \tag{7}$$

while their initial values are given by $\xi(0) = \vartheta(0) = 0, \rho(0) = 1$. For $\epsilon = 0$ the system (7) can be solved explicitly, and one finds that $\vartheta(\tau) = -\arcsin\tanh\frac{\tau}{2}$. Since the solution of (7) depends smoothly on the parameter ϵ, we may conclude the following.

Lemma 1. *There is a $t_2 > 0$ such that for all sufficiently large R one has $T(R) > t_2/R$, while, at $t = t_2/R$, one will have $-\pi/3 \le \theta_R \le -\pi/6$, $x_R = \mathcal{O}(1/R)$ and $r_R = R - \mathcal{O}(1/R^2)$.*

As we noted before, for $t \le t_1(R)$ we have $0 > \theta_R(t) > -\pi$, so that γ_R is a graph. We shall now estimate how far it can get from the r-axis.

Define $\alpha = \frac{\pi}{2} + \theta$, and regard α as a function of r. Then we have

$$\begin{aligned} \frac{d\alpha}{dr} &= \frac{\theta'}{r'} = \frac{x}{2} + \left(\frac{r}{2} - \frac{n-1}{r}\right)\tan\alpha \\ &\ge \left(\frac{r}{2} - \frac{n-1}{r}\right)\tan\alpha. \end{aligned} \tag{8}$$

Let $R_1 = r_R(t_1)$, and integrate this inequality from any $r < R_1$ with $\alpha(r') > 0$ for $r < r' < R_1$, to R_1. This shows that

$$\frac{\sin\alpha(R_1)}{\sin\alpha(r)} \ge e^{(R_1^2 - r^2)/2}\left(\frac{r}{R_1}\right)^{n-1},$$

i.e. we find that for all r such that $\theta \in (-\frac{\pi}{2}, 0)$ on (r, R_1) we have:

$$\sin\alpha(r) \le \left(\frac{R_1}{r}\right)^{n-1} e^{-(R_1^2 - r^2)/2} \sin\alpha(R_1). \tag{9}$$

In particular we have $\sin\alpha(r) \le \sin\alpha(R_1) \le \frac{1}{2}\sqrt{3}$ for all $r \in (\frac{1}{R}, R_1)$ with $\theta \in (-\frac{\pi}{2}, 0)$ on (r, R_1). For such r's one therefore has $\tan\alpha \le 2\sin\alpha$. Since $\tan\alpha = -f_R'(r)$, this implies that for all $1/R < r < R_1$ with $\theta \in (-\frac{\pi}{2}, 0)$ on (r, R_1) one has
(10)

$$f_R(r) \le f_R(R_1) + 2\int_r^{R_1} \left(\frac{R_1}{r}\right)^{n-1} e^{-(R_1^2 - r^2)/2} dr \le \frac{C}{R} \qquad (R \to \infty).$$

This inequality gives us an estimate for the first maximum of $f_R(r)$ which one might encounter if one decreases r, starting at $r = R_1$. We had already observed that f_R can have at most one maximum, so that the estimate (10) implies that we have proved

Lemma 2. *For $0 < t < t_1$ one has $x_R(t) \le C/R$ or $r_R(t) < 1/R$, if R is large enough. The constant C does not depend on R.*

The next step in the proof of our theorem is to show that the geodesic γ_R will intersect the r-axis, if R is large enough.

Lemma 3. *For large enough R one has $t_1(R) < \infty$, while $x_R(t_1) = 0$, and $r_R(t_1) \ge 1/R$.*

Proof. Suppose not. Then there exists a sequence $R_n \uparrow \infty$ for which $x_{R_n}(t) > 0$ at least as long as $r_{R_n}(t) > 1/R_n$. In this situation the functions $f_{R_n}(r)$ are defined for $1/R_n < r < R_n$, on which interval they satisfy

$$\frac{f''(r)}{1 + f'(r)^2} + \left(\frac{n-1}{r} - \frac{r}{2}\right) f'(r) + \frac{f(r)}{2} = 0. \tag{11}$$

Moreover it follows from Lemma 2 that the $f_{R_n}(r)$ are bounded by $0 < f_{R_n}(r) < C/R_n$. Since the geodesic γ_{R_n} gets arbitrarily close to the r-axis, its tangents also must converge to the r-axis; if for each n there were some point $p_n \in \gamma_{R_n}$ whose tangent made an angle $\theta \in (-\pi + \delta, -\delta)$ for some constant $\delta > 0$, and if these points remained in a compact domain in the upper half plane, then one could extract a convergent subsequence, $p_{n_j} \to p_*$. For large n_j the geodesic $\gamma_{R_{n_j}}$ would have to be close to the geodesic through p_* with the same tangent direction, and hence would have to intersect the r axis somewhere near p_*. We are assuming that this doesn't happen, so that this argument shows us that both $f_{R_n}(r)$ and $f_{R_n}'(r)$ converge uniformly to zero, on compact intervals in $(0, \infty)$.

Since the f_{R_n}'s are solutions of (11), and since we know that f_{R_n}' converges uniformly to zero, it follows from the positivity of the f_{R_n} that there is a constant $C < \infty$ for which one has $|f'_{R_n}(1)| \leq C|f_{R_n}(1)|$. It follows that one can now extract a subsequence $n_1 < n_2 < n_3 < \cdots$ of the integers for which

$$\frac{f_{R_{n_j}}(r)}{f_{R_{n_j}}(1)} \to g(r), \qquad (0 < r < \infty)$$

where the convergence is in C^2 on any compact interval in $(0, \infty)$.

The limit function $g(r)$ is a nontrivial, positive solution of the linearized equation of (11), i.e. of

$$g''(r) + \left(\frac{n-1}{r} - \frac{r}{2}\right) g'(r) + \frac{g(r)}{2} = 0 \tag{12}$$

on $0 < r < \infty$. A short computation will reveal that $h(r) = e^{r^2/8} g(r)$ satisfies

$$h''(r) + \frac{n-1}{r} h'(r) + \left(\frac{n}{4} - \frac{r^2}{16}\right) h(r) = 0. \tag{13}$$

Up to a constant term the differential operator in this equation is the radial part of the Harmonic Oscillator on $\mathbf{R}^n$, so that it is easily verified that (13) has no positive solutions. But this contradicts the fact that $h(r) > 0$.

Q. E. D.

On the other hand, we have seen that the circle $x^2 + r^2 = 2n$ is a geodesic for the metric (5), so that $t_1(R) = T(R)$ at $R = \sqrt{2n}$. It follows that there is a smallest $R_* \geq \sqrt{2n}$ for which $t_1(R) < \infty$ and $x_R(t_1) = 0$ holds whenever $R > R_*$.

Lemma 4. $\inf_{R_* < R \leq R_* + 1} r_R(t_1(R)) > 0$.

Proof. We argue by contradiction again. Assume that $r_{R_n}(t_1(R_n)) \downarrow 0$ for some sequence $R_n \downarrow R_*$. We reparametrize the geodesics so that $r\frac{d}{dt} = \frac{d}{d\tau}$. In other words, we introduce a new time variable which is related to t via

$$\tau = \int^t \frac{dt}{r(t)}.$$

In this new time variable τ the equations (6) become regular at $r = 0$. Indeed one finds:

$$\begin{cases} \dfrac{dx}{dt} = r\cos\theta, \qquad \dfrac{dr}{dt} = r\sin\theta, \\ \dfrac{d\theta}{dt} = \left(n - 1 - \dfrac{r^2}{2}\right)\cos\theta + \dfrac{xr}{2}\sin\theta. \end{cases} \tag{14}$$

This system has a line of fixed points, $\ell = \{r = 0, \theta = -\frac{\pi}{2}\}$. By linearizing (14) one finds that this line is normally hyperbolic.

At $t = t_1(R_n)$ we have $-\frac{\pi}{2} > \theta(t_1(R_n)) > -\pi$, while

$$\frac{d\theta}{d\tau} = (n-1)\cos\theta + \mathcal{O}(|x|^2 + |r|^2), \qquad \frac{dx}{d\tau}, \frac{dr}{d\tau} = \mathcal{O}(|x| + |r|).$$

This means that just before the geodesic γ_{R_n} hits the r-axis, one has $\theta \approx -\frac{\pi}{2}$, with $x, r = o(1)$. Thus the curve Γ_{R_n} converges to the stable manifold of the point $(x, r, \theta) = (0, 0, -\frac{\pi}{2})$ on the line ℓ. But this stable manifold is exactly the "planar solution", i.e. the r-axis. This cannot happen, since γ_{R_n} also converges to γ_{R_*}, a geodesic distinct from the r-axis.

Q. E. D.

This lemma shows us that the γ_R are bounded away from the x axis, as $R \downarrow R_*$. In the following lemma we also obtain a bound in the x direction for the γ_R.

Lemma 5. $\sup(x_R(t) : 0 < t < t_1(R), R_* < R \le R_* + 1) < \infty$.

Proof. If the lemma is not true, then for some sequence $R_n \downarrow R_*$ the maximal value $\xi_n = \max\{x_R(t) : 0 < t < t_1\}$ becomes unbounded. The geodesic $\gamma_{R_n}([0, t_1(R_n)])$ is a graph $x = f(r)$, with one maximum, and hence it can also be represented as the union of two graphs $r = g_{n,\pm}(x)$ where $0 \le x \le \xi_n$, and where $g_{n,-}(x) < g_{n,+}(x)$ for $0 \le x < \xi_n$. Moreover, we have $\delta \le g_{n,\pm}(x) \le R_* + 1$ for all n, and all $x \le \xi_n$. Finally, the $g_{n,\pm}$'s are strictly monotone, and they satisfy the following differential equation:

$$\frac{g''(x)}{1 + g'(x)^2} - \frac{x}{2} g'(x) + \frac{n-1}{g(x)} - \frac{g(x)}{2} = 0. \tag{15}$$

By studying the asymptotic form of the equations (6) for large x and bounded r, one finds that the slopes $g'_{n,\pm}(x)$ must be uniformly bounded on $[0, \xi_n - 1]$. Indeed, if one introduces a new time variable, s, with $ds = x\,dt$, then the equations (6) imply that

$$\begin{aligned} x_s, r_s &= \mathcal{O}(\frac{1}{x}) \\ \theta_s &= \frac{1}{2}\sin\theta + \mathcal{O}(\frac{1}{x}) \end{aligned} \tag{16}$$

hold in the region $\delta \le r \le R_* + 1$. If one neglects the $\mathcal{O}(\frac{1}{x})$ terms, then these equations can be solved explicitely: x and r are constant, and $\theta(s) = -\frac{\pi}{2} - \operatorname{arcsin}\tanh\frac{s}{2}$. If $|g'_{n,\pm}(x)|$ were large at some point $p \in \gamma_{R_n}$, then the corresponding angle θ would be close to $-\frac{\pi}{2}$, and it would follow from the equations (16) that the geodesic γ_{R_n} must have a vertical tangent at a distance $\mathcal{O}(\frac{1}{x})$ from the point p. If we choose p so that its x coordinate does not exceed $\xi_n - 1$, then this is impossible, since the geodesic is a graph $r = g_{n,\pm}(x)$ near p.

Thus we have a bound for the derivatives of the $g_{n,\pm}$'s. It follows from (15) that we then also have a bound for the second, and by induction, all higher order derivatives of $g_{n,\pm}$. After passing to a subsequence, if necessary, one may assume that the $g_{n,\pm}$ converge in $C^\infty_{\mathrm{loc}}(\mathbf{R}_+)$ to two functions $g_\pm(x)$, both of which again satisfy (15). These two functions must be different, for their values at $x = 0$ are $g_+(0) = R_* \geq \sqrt{2n}$, and $g_-(0)$, respectively. If $g_- + (0) = g_-(0)$, then the geodesics γ_{R_n} would be trapped between the horizontal lines $r = g_{n,-}(0)$ and $r = g_{n,+}(0)$, which converge to the line $r = g_+(0)$. This limit line would then also have to be a geodesic, but the only horizontal line which is a geodesic is the line $r = \sqrt{2(n-1)}$. Thus we have $g_-(x) < g_+(x)$ for all $x \geq 0$.

We can write (15) as $g''(x) = F(x, g(x), g'(x))$, where

$$F(x,g,p) = (1+p^2)\left(\frac{xp}{2} - \frac{n-1}{g} + \frac{g}{2}\right)$$

satisfies

$$\frac{\partial F}{\partial g}(x,g,p) > 0,$$
$$\frac{\partial F}{\partial p}(x,g,p) = \frac{x}{2}(1+3p^2) - 2p\left(\frac{n-1}{g} - \frac{g}{2}\right) \geq \frac{x}{4} - C,$$

for all $\delta < g < R_* + 1$, and $p \in \mathbf{R}$, and for a sufficiently large constant C. Using the mean value theorem one then finds that the difference $z(x) = g_+(x) - g_-(x)$ satisfies a linear equation of the form $z''(x) - M(x)z'(x) - N(x)z(x) = 0$, in which $M(x) \geq \frac{x}{4} - C$ and $N(x) \geq 0$. Since $z(x) > 0$ this implies the differential inequality $z''(x) - M(x)z'(x) \geq 0$, which one can integrate once, on the interval (x_0, x_1). The result is:

$$z'(x_0) \leq \exp\left\{-\int_{x_0}^{x_1} M(x)dx\right\} z'(x_1) \leq C' e^{-x_1^2/8} z'(x_1).$$

Using our bounds on $g'_\pm(x)$ and choosing x_1 arbitrarily large, this leads us to the conclusion that $z'(x_0) = 0$, for all $x_0 \geq 0$. Thus (15) does not admit two ordered solutions $g_- < g_+$ on $\mathbf{R}_+$. Since our original assumption led us to two such solutions, it must have been false.

Q. E. D.

So far we have found that the geodesics γ_R stay away from the x axis, and remain uniformly bounded as $R \downarrow R_*$. Therefore the limiting geodesic $\gamma_* = \gamma_{R_*}$ begins and ends on the r-axis, i.e. it goes from $(0, R_*)$ to $(0, r_*)$, where $r_* = r_{R_*}(t_1(R_*))$.

We now claim that $\theta_* = \theta_{R_*}(t_1(R_*)) = -\pi$. We know that $\theta_* \geq -\pi$; if one had strict inequality, then one could vary the height R near R_*, and still obtain a geodesic γ_R, which, in the first quadrant, is a graph $x = f_R(r)$, and which hits the r-axis in finite time. Since one would be able to do this for $R < R_*$, but close to R_*, this would contradict the minimality of R_*. Thus $\theta_* = -\pi$, as claimed. This also completes the proof of the main theorem.

The pinching neck in the flow by mean curvature problem.

In this section we'll show how the existence of the shrinking torus leads to the formation of singularities in finite time, for some solutions of the flow by mean curvature problem. We shall restrict our discussion to the most easily visualized case, i.e. to the motion of surfaces in three dimensional space.

We shall consider a solution to the flow by mean curvature problem whose initial value Σ_0 is the surface suggested in Figure 4. The surface is the image of an embedding $X_0 : S^2 \to \mathbf{R}^3$. The relevant features of Σ_0 are:

(i) it encloses two spheres of radius $R > 0$,

(ii) it has a "neck" which is circled by one of the self similar shrinking tori, whose existence we have just established.

Figure 4.

One can let the two spheres and the torus flow according to their mean curvature; each will shrink in a self similar way to its center. We shall assume that the radius R of the two spheres is so large that the little torus will shrink to a point before the spheres do so. Denote the time at which the torus disappears by T_*.

Let $X : S^2 \times [0, T) \to \mathbf{R}^3$ be the maximal solution of (1) with initial value X_0. Then one must have $T \leq T_*$. Indeed, it follows from the maximum principle for parabolic equations that the four solutions (X, the two spheres and the torus) remain disjoint for as long as they are defined (see

[Br, ES]). Thus if the smooth solution X existed for $t > T_*$, then at time $t = T_*$ the surface $X(S^2, T_*)$ still encloses the two spheres, and it must have a neck which is circled by the torus. But this torus just shrank to a point, so the neck must have zero diameter at some point, which shows that the solution X is singular at time T_*.

The final conclusion is that the solution corresponding to the initial value depicted in Figure 4 becomes singular before it can become convex, so that we have one more proof that Grayson's theorem cannot be extended to higher dimensions.

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