A Variational Interpretation of Melnikov's Function and Exponentionally Small Separatrix Splitting

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1 Introduction. This note is about the exponentially small separatrix splitting which occurs when one studies the separatrices of maps of "standard type"

$$\Phi(u,v) = (u+\varepsilon, v+\varepsilon f_0(u+\varepsilon v)),$$

where f_0 is an entire function, or when one considers the Poincaré–map associated with the ODE

$$u''(t) = F(t/\varepsilon, u(t)) \tag{1.1}$$

for small values of $\varepsilon > 0$, and nonlinearities F with $F(t+1,u) = F(t,u)$ which are analytic in the u variable.

We recall that the Poincaré–map Φ_ε is defined in terms of the first order system

$$u' = v, v' = F(t/\varepsilon, u)$$

which is equivalent to the second order ODE (1.1); Φ_ε sends $(u(0), v(0))$ to $(u(\varepsilon), v(\varepsilon))$, where $(u(t), v(t)), (0 \le t \le \varepsilon)$ is a solution of (1.1).

For small $\varepsilon > 0$ the theory of averaging tells us that we may regard (1.1) as a "small" perturbation of the averaged equation

$$u'' = F_0(u) \tag{1.2}$$

with $F_0(u) = \int_{\mathbb{T}} F(\tau, u) d\tau$.

If the Poincaré map associated with (1.2) has a hyperbolic fixed point with a homoclinic orbit, then one expects the same to be true[1] for the perturbed Poincaré map Φ_ε ($\varepsilon << 1$). One also expects the homoclinic orbit of perturbed map to come from a transverse intersection of the invariant manifolds through the hyperbolic fixed point. Melnikov's method allows one to verify this for smooth perturbations of (1.2) with fixed period, e.g. equations of the form $u'' = F_0(u) + \mu g(t,u)$. However, it has been observed that the method does not apply directly to (1.1). Holmes, Marsden and Scheurle[2] were the first to try to adjust Melnikov's method to the averaging situation. They gave an asymptotic expression for the separatrix splitting if F is of the form $F(\tau, u) = \sin u + \delta\varepsilon^p g(t)$, with g periodic, p sufficiently

[1] See chapter 4 of [GH83] for a discussion of averaging.
[2] [HMS88]

large, and δ a small parameter. This result was later improved by various authors, the best result to date being due to Delshams, Teresa and Seara[3].

In section 2 we give a variational interpretation of the Melnikov function, and in the subsequent sections show how this interpretation can be adapted to study the homoclinic orbits of (1.1). Like Holmes et.al. we only get an upper bound for the size of the splitting in the most general setting, while we only get transverse homoclinic intersections for a special nonlinearity, $F(\tau, u) = u - {}^3/_2 u^2 + \delta\varepsilon^{10} H'(\tau)$, with $H(\tau)$ periodic, and δ a small parameter. For this particular example the variational approach is an improvement on the results of Holmes et.al. but fails to give the result of Delshams et.al.

One advantage the variational point of view may have over others, is that it can easily be generalized, to find entire solutions of elliptic PDE's such as

$$\Delta u = F_0(u) + \mu g(x, u(x)), \qquad u(\infty) = 0, \tag{1.3}$$

where $g(x, u)$ is periodic in the x-variable; given a nontrivial solution $U(x)$ of the spatially homogeneous equation $\Delta u = F_0(u)$ which vanishes at $x = \infty$, the analysis in section 2 allows one to find solutions of (1.3) close to some translate $U(x + \vartheta)$ of $U(x)$.

Although there is no obvious Poincaré–map in this situation, one can still show[4] that nondegenerate solutions of (1.3) generate many more solutions of (1.3), much in the same way that a transverse homoclinic point of the Poincaré–map generates an abundance of homoclinic orbits.

The two main examples we have in mind throughout the paper are a forced Duffing equation

$$u'' - F_0(u) = \delta g(t), \tag{1.4}$$

and a "kicked anharmonic oscillator"

$$u''(t) = \varepsilon \sum_{j \in \mathbb{Z}} \delta(t - j\varepsilon) \cdot F_0(u(t)) \tag{1.5}$$

with $F_0(u) = u - {}^3/_2 u^2$.

In the second example the equation is to be interpreted in the sense of distributions: a solution is a Lipschitz function whose second distributional derivative satisfies (1.5). In fact, solutions will be piecewise linear, and their values $u_j = u(j\varepsilon)$ satisfy the recurrence relation

$$u_{j+1} - 2u_j + u_{j-1} = \varepsilon^2 F_0(u_j). \tag{1.6}$$

One easily verifies that the Poincaré–map Φ_ε is given by the standard type map $(u, v) \mapsto (u + \varepsilon v, v + \varepsilon F_0(u + \varepsilon v))$.

In section 3 we introduce a large class of nonlinearities F which includes both of these examples.

2 A variational account of the Melnikov function. We assume in this section that $\varepsilon = 1$, and that the nonlinearity F is of the form $F(t, u) = F_0(u) + \mu g(t, u)$, where g is some smooth function, μ is small, and F_0 satisfies

$$F_0(0) = 0, \quad F_0'(0) > 0. \tag{2.1}$$

This last condition implies that the origin is a hyperbolic fixed point for the local flow Ψ_t generated by the system $u' = v, v' = F_0(u)$.

The potential energy associated with F_0 is given by

$$V_0(u) = -\int_0^u F_0(\omega)\,d\omega.$$

[3] See [DTS91] and the references given there.

[4] See [A86].

We shall assume that $V_0(u) < 0$ for some $u > 0$, and that $V_0'(\alpha) = -F_0(\alpha) < 0$ where α is the smallest positive root of $V_0(\alpha) = 0$. Under this assumption the stable and unstable manifolds W^u, W^s of the origin coincide; they are parametrized by $(U(t), U'(t))$ where U is the unique positive and even solution of

$$U'' = F_0(U), \qquad U(\pm\infty) = 0. \tag{2.2}$$

Consider the Poincaré map Φ_μ of the perturbed system $u' = v, v' = F(\mu, t, u)$. If μ is small Φ_μ will have a hyperbolic fixed point $\mathcal{O}_\mu$ near the origin, whose stable and unstable manifolds we denote by W^s_μ, W^u_μ. Since Φ_μ depends smoothly on μ, the fixed point $\mathcal{O}_\mu$ as well as the W^s_μ, W^u_μ vary smoothly with μ.

For most perturbations $g(t,u)$ the invariant manifolds W^s_μ, W^u_μ will not coincide when $\mu \neq 0$. Melnikov's method was designed to compute the separation between the invariant manifolds for small values of μ, and in particular, to find the transverse intersections in $W^s_\mu \cap W^u_\mu$.

Figure 1 Here

It is a commonplace[5] to remark that these transverse intersections are of interest since they are known to be a cause of complicated dynamics of the Poincaré map Φ_μ.

We shall now proceed to describe a variational method which produces a result equivalent to Melnikov's. To begin with we construct a periodic solution which corresponds to the hyperbolic fixed point $\mathcal{O}_\mu$ by applying the implicit function theorem to the map $\mathcal{F} : \mathbb{R} \times C^2(\mathbb{T}) \to C^0(\mathbb{T})$ given by

$$\mathcal{F}(\mu, p) = p'' - F_0(p) - \mu g(t, p).$$

Here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $C^k(\mathbb{T})$ is the space of k times continuously differentiable functions $u(t)$ with $u(t+1) \equiv u(t)$.

We have $\mathcal{F}(0,0) = 0$ while $\mathrm{d}_p\mathcal{F}(0,0)$, the derivative of $\mathcal{F}$ w.r.t. p, is given by $\mathrm{D}^2 - F_0'(0)$; since $F_0'(0) > 0$ the operator $\mathrm{d}_p\mathcal{F}(0,0)$ has a bounded inverse from $C^0(\mathbb{T})$ to $C^2(\mathbb{T})$, and we have a smooth branch of solutions $p(\mu, \cdot) \in C^2(\mathbb{T})$ of $\mathcal{F}(\mu, p) = 0$ with $p(0,t) \equiv 0$. The fixed point $\mathcal{O}_\mu$ is now given by $(p(\mu, 0), p'(\mu, 0))$.

Homoclinic orbits of Φ_μ correspondend to solutions $u(t)$ of $u'' = F(\mu, t, u)$ which are defined for all $t \in \mathbb{R}$, and which are asymptotic to the small solution $p(\mu, t)$ as $t \to \pm\infty$. To find such solutions we substitute $u(t) = v(t) + p(\mu, t)$ and obtain the following equation for v:

$$v'' = \hat{F}(\mu, t, v(t)), \qquad v(\pm\infty) = 0, \tag{2.3}$$

where

$$\begin{aligned}\hat{F}(\mu, t, v) &= F(\mu, t, p + v) - p'' \\ &= F_0(p + v) - F_0(p) + \mu\,\{g(t, p + v) - g(t, p)\}\end{aligned}$$

with $p = p(\mu, t)$, and $' = \partial/\partial t$. The corresponding potential energy is given by

$$\hat{V}(\mu, t, v) = \int_0^v \hat{F}(\mu, t, \omega) d\omega;$$

it satisfies $|\hat{V}(\mu, t, v)| \leq Cv^2$ for small v, and hence the functional

$$\mathcal{A}_\mu(v) = \mathcal{A}(\mu, v) = \int_{\mathbb{R}} \left({}^1\!/_2 v'(t)^2 - \hat{V}(\mu, t, v(t)) \right) dt \tag{2.4}$$

is well defined for $v \in H^1(\mathbb{R})$.

[5] See [Mo73, HG84] and the references given there.

2.1. Lemma. *Critical points of $\mathcal{A}_\mu$ are exactly the solutions of (2.3), and hence they correspond to the homoclinic orbits of Φ_μ, i.e. to the intersections of W^s_μ and W^u_μ.*

For small μ a critical point of $\mathcal{A}_\mu$ is nondegenerate if and only if the corresponding intersection of W^s_μ and W^u_μ is transverse.

Proof. The first statement holds since (2.3) is the Euler-Lagrange equation for $\mathcal{A}_\mu$.

Concerning the connection between nondegeneracy and transversality we remark first of all that the Poincaré–maps Φ_μ and $\hat{\Phi}_\mu$, where the latter is derived from $w'' = \hat{F}(\mu, t, w)$, are conjugate. The conjugation is provided by the translation

$$\tau : (u_0, u_0') \mapsto (v_0, v_0') = (u_0 + p(\mu, 0), u_0' + p_t(\mu, 0)) .$$

Thus if $P \in W^u_\mu \cap W^s_\mu$, then $\tau(P) \in \hat{W}^u_\mu \cap \hat{W}^s_\mu$, and $\hat{W}^u_\mu$ and $\hat{W}^s_\mu$ intersect transversally at $\tau(P)$ *iff* W^u_μ and W^s_μ do so at P. We may therefore consider $\hat{W}^u_\mu$ and $\hat{W}^s_\mu$ instead of W^u_μ and W^s_μ.

Let $v \in H^1$ be a critical point of $\mathcal{A}_\mu$. Then $v \in C^\infty$, and $P = (v(0), v'(0))$ is the corresponding intersection of $\hat{W}^u_\mu$ and $\hat{W}^s_\mu$. The second derivative of $\mathcal{A}_\mu$ at v is given by

$$\mathrm{d}^2\mathcal{A}_\mu(v) \cdot (\varphi, \psi) = \langle \mathrm{L}\varphi, \psi \rangle,$$

where $\mathrm{L} : H^1 \to H^{-1}$ is the differential operator

$$\mathrm{L} = -\mathrm{D}^2 + Q(t); \qquad Q(t) =_{\mathrm{def}} \frac{\partial \hat{F}}{\partial u}(\mu, t, v(t)),$$

and where $\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \varphi\psi$.

When μ is small $p(\mu, t)$ is also small, so it follows from

$$\hat{F}_u(\mu, t, v) = F_0'(p(\mu, t) + v) + \mu g_u(t, p(\mu, t) + v),$$

$F_0'(0) > 0$, and $v(\pm\infty) = 0$ that for small μ

$$\liminf_{t \to \pm\infty} Q(t) > 0. \tag{2.5}$$

Hence L is Fredholm with index zero for small μ.

Indeed, $\mathrm{L}_0 = -\mathrm{D}^2 + Q_0(t)$ with $Q_0(t) = \hat{F}_u(\mu, t, 0)$ is invertible, since $\inf Q_0(t) > 0$; $\mathrm{L} - \mathrm{L}_0$ is given by multiplication with $Q(t) - Q_0(t)$, which vanishes at $t = \pm\infty$ and hence is a compact operator from H^1 to H^{-1}; so L is indeed Fredholm.

By definition the critical point v will be nondegenerate *iff* $\mathrm{L} = \mathrm{d}^2\mathcal{A}_\mu(v)$ is invertible, which, due to L's Fredholmness, will be the case *iff* L is injective. The nullspace of L consists of those $y \in H^1$ which satisfy

$$y'' = Q(t)y. \tag{2.6}$$

At $t = \pm\infty$ $Q(t)$ is bounded away from zero, so solutions of (2.6) as well as their derivatives either grow or decay exponentially. Hence there are two solutions $y_\pm(t)$ of (2.6) with $y_+(+\infty) = 0$ and $y_-(-\infty) = 0$, respectively.

Since (2.6) is the variational equation of (2.3), the vectors $\xi_+ = (y_+(0), y_+'(0))$ and $\xi_- = (y_-(0), y_-'(0))$ span the tangent spaces $T_P W^s_\mu$ and $T_P W^u_\mu$, respectively.

If the intersection is not transverse, then $\xi_- = \lambda\xi_+$, and hence $y_- = \lambda y_+$; in this case y_+ not only vanishes at $t = \infty$, but also at $t = -\infty$. Condition (2.5) implies that bounded solutions of (2.6) decay exponentially as $t \to \pm\infty$, so y_+ belongs to the nullspace of L, and v is a degenerate fixed point.

Conversely, if v is degenerate, then the $y_\pm$ are multiples of each other, whence the $\xi_\pm$ are also multiples of each other; i.e. the intersection at P is nontransverse.

Q. E. D.

For $\mu = 0$ we have a curve $\Sigma \subset H^1(\mathbb{R})$ of critical points of $\mathcal{A}_\mu$, consisting of all translates $U_\vartheta(t) = U(\vartheta + t), \vartheta \in \mathbb{R}$ of U. All these critical points are degenerate, but they are nondegenerate in the direction transverse to the curve Σ. Indeed, define

$$E_\vartheta^r = \left\{ v \in H^r(\mathbb{R}) : \int_{\mathbb{R}} v(t) {U_\vartheta}'(t) dt = 0 \right\}, \qquad r = \pm 1.$$

Using $U_\vartheta(\pm\infty) = 0$ one easily verifies that $U_\vartheta \in E_\vartheta^1$. Since ${U_\vartheta}'$ spans the tangent space to the curve Σ, E_ϑ^1 intersects Σ transversally at U_ϑ.

Figure 2 here

The second derivative of $\mathcal{A}_0 = \mathcal{A}(0, \cdot)$ at U_ϑ is given by $\mathrm{d}^2\mathcal{A}_0(U_\vartheta) \cdot (\varphi, \psi) = \langle \mathrm{A}_\vartheta \varphi, \psi \rangle$, where $\mathrm{A}_\vartheta : H^1 \to H^{-1}$ is the operator $\mathrm{A}_\vartheta = -\mathrm{D}^2 + F_0'(U_\vartheta(t))$.

2.2. Lemma. *A_ϑ is a Fredholm operator of index zero. Its kernel is spanned by ${U_\vartheta}'$, and its range is E_ϑ^{-1}.*

Proof. That A_ϑ is Fredholm follows from the same arguments we used in the previous lemma.

The kernel of A_ϑ clearly contains ${U_\vartheta}'$ – just differentiate (2.2) – and by considering the growth or decay of solutions of $y''(t) = F_0'(U_\vartheta(t)) y(t)$ at $t = \pm\infty$ one finds that the kernel of A_ϑ can be at most one–dimensional. Hence it must be spanned by ${U_\vartheta}'$.

It follows that the range of A_ϑ has codimension one, while integration by parts shows that $\mathrm{A}_\vartheta(H^1) \subset E_\vartheta^{-1}$. Hence $\mathrm{A}_\vartheta(H^1) = E_\vartheta^{-1}$.

Q. E. D.

2.3. Corollary. *U_ϑ is a Morse critical point of $\mathcal{A}_0 | E_\vartheta^1$.*

It follows that the critical point U_ϑ of $\mathcal{A}_0 | E_\vartheta^1$ will persist under small perturbations, so that we obtain a smooth family of critical points $v_{\vartheta,\mu} \in E_\vartheta^1$ which is defined for small μ and $\vartheta \in \mathbb{R}$.

If $\mathcal{A}_\mu$ has a critical point v_* near any of the U_ϑ's then $v_* = v_{\vartheta,\mu}$ for some $\vartheta \in \mathbb{R}$, but the converse is not true: not every $v_{\vartheta,\mu}$ is a critical point of $\mathcal{A}_\mu$. To determine which of the $v_{\vartheta,\mu}$ are critical points we observe that, since $v_{\vartheta,\mu}$ is a critical point of $\mathcal{A}_\mu$ subject to the constraint $\langle {U_\vartheta}', v \rangle = 0$, we have

$$\mathrm{d}\mathcal{A}_\mu(v_{\vartheta,\mu}) = \lambda {U_\vartheta}',$$

where $\lambda = \lambda(\mu, \vartheta)$ is a Lagrange multiplier. This obviously implies:

2.4. Proposition. *$v_{\vartheta,\mu}$ is a critical point of $\mathcal{A}_\mu$ iff $\lambda(\mu, \vartheta) = 0$.*

Rather than considering $\lambda(\mu, \vartheta)$ we introduce

$$a(\mu, \vartheta) = \mathcal{A}_\mu(v_{\vartheta,\mu}),$$

and compute

$$\begin{aligned} \frac{\partial a}{\partial \vartheta} &= \left\langle \mathrm{d}\mathcal{A}_\mu(v_{\vartheta,\mu}), \frac{\partial v_{\vartheta,\mu}}{\partial \vartheta} \right\rangle \\ &= \lambda(\mu, \vartheta) \left\langle {U_\vartheta}', \frac{\partial v_{\vartheta,\mu}}{\partial \vartheta} \right\rangle. \end{aligned}$$

For $\mu = 0$ we have $v_{\vartheta,\mu} = U_\vartheta$, and hence $\frac{\partial v_{\vartheta,\mu}}{\partial \vartheta} = {U_\vartheta}'$, so that we find

$$\lambda(\mu, \vartheta) = \frac{\partial a}{\partial \vartheta}(\mu, \vartheta)\left(\Gamma + o(1)\right), \qquad (\mu \to 0),$$

with $\Gamma = \int_{\mathbb{R}} U'(t)^2 dt > 0$. Hence for small μ the zeroes of $\lambda(\mu, \cdot)$ and $a_\vartheta(\mu, \cdot)$ coincide.

The functional $\mathcal{A}(0, v)$ is invariant under translations of v, i.e. under the substitution $v(t) \mapsto v(t + \vartheta)$, so $a(0, \vartheta) = \mathcal{A}(0, U_\vartheta)$ does not depend on ϑ. Therefore $a_\vartheta(0, \vartheta) \equiv 0$, and $a_\vartheta(\mu, \vartheta)/\mu$ is a well defined smooth function for small μ; we have

$$a_\vartheta(\mu, \vartheta) = \mu \frac{\partial^2 a}{\partial \vartheta \partial \mu}(0, \vartheta) + O(\mu^2), \qquad (\mu \to 0).$$

The second derivative can be computed directly:

$$\begin{aligned}
\frac{\partial^2 a}{\partial \vartheta \partial \mu}(0, \vartheta) &= \frac{\partial}{\partial \vartheta}\left(\frac{\partial}{\partial \mu}\mathcal{A}_\mu(v_{\mu,\vartheta})\right) \\
&= \frac{\partial}{\partial \vartheta}\left(\frac{\partial \mathcal{A}}{\partial \mu}(0, U_\vartheta) + \mathrm{d}\mathcal{A}_0(U_\vartheta)\frac{\partial v_{\mu,\vartheta}}{\partial \mu}\right) \\
&= \frac{\partial}{\partial \vartheta}\left(\frac{\partial \mathcal{A}}{\partial \mu}(0, U_\vartheta)\right) \qquad (\text{since } \mathrm{d}\mathcal{A}_0(U_\vartheta) = 0) \\
&= -\frac{\partial}{\partial \vartheta}\int_{\mathbb{R}} \frac{\partial \hat{V}}{\partial \mu}(0, t, U_\vartheta(t))dt \qquad (\text{by } (2.4)) \\
&= \int_{\mathbb{R}} \frac{\partial \hat{F}}{\partial \mu}(0, t, U_\vartheta(t)){U_\vartheta}'(t)dt.
\end{aligned}$$

Using our definition of $\hat{F}$ we find that

$$\frac{\partial \hat{F}}{\partial \mu}(0, t, v) = \left(F_0'(v) - F_0'(0)\right) q(t) + g(t, v) - g(t, 0),$$

where $q(t) = \frac{\partial p}{\partial \mu}(0, t)$ is obtained by solving

$$q''(t) - F_0'(0)q(t) = g(t, 0), \qquad q(t+1) \equiv q(t).$$

This identity implies that

$$\begin{aligned}
\int_{\mathbb{R}} g(t, 0){U_\vartheta}'(t)dt &= \int_{\mathbb{R}} \left({U_\vartheta}''' - F_0'(0){U_\vartheta}'\right) q(t)dt \\
&= \int_{\mathbb{R}} {U_\vartheta}'(t)\left(F_0'(U_\vartheta(t)) - F_0'(0)\right) q(t)dt;
\end{aligned}$$

from which we get the following formula for $a_{\mu\vartheta}$

$$a_{\mu\vartheta}(0, \vartheta) = \int_{\mathbb{R}} g(t, U_\vartheta(t)){U_\vartheta}'(t)dt.$$

This last integral is precisely the Melnikov function. In general we can now easily prove:

2.5. Theorem (Melnikov). *Assume that the Melnikov function $\mathcal{M}(\vartheta) = a_{\mu\vartheta}(0,\vartheta)$ has a simple zero at $\vartheta = \vartheta_0$. Then there exists a smooth branch of nondegenerate critical points $w_\mu = v_{\mu,\vartheta(\mu)}$ of $\mathcal{A}_\mu$ with $w_0 = U_{\vartheta_0}$. In particular, for small enough μ the stable and unstable manifolds W^s_μ, W^u_μ have a transverse intersection at $P_\mu = (w_\mu(0)+p(\mu,0), w'_\mu(0)+p_t(\mu,0))$.*

Proof. We have just shown that $b(\mu,\vartheta) = a_\vartheta(\mu,\vartheta)/\mu$ is a smooth function, defined for small μ, with $b(0,\vartheta) = \mathcal{M}(\vartheta)$. The implicit function theorem provides us with a smooth branch of zeroes $\vartheta(\mu)$ of $b(\mu,\vartheta)$, hence of $a_\vartheta(\mu,\vartheta)$. Defining $w_\mu = v_{\mu,\vartheta(\mu)}$ as we did, we get a branch of critical points.

To show that the w_μ are nondegenerate we consider the splitting $H^1 = E^1_{\vartheta(\mu)} \oplus [U'_{\vartheta(\mu)}]$; w.r.t. this splitting we can represent the second derivative of $\mathcal{A}_\mu$ at w_μ by

$$\mathrm{d}^2\mathcal{A}_\mu(w_\mu) = \begin{pmatrix} \mathrm{A}_{\vartheta(\mu)}|E^1_{\vartheta(\mu)} + O(\mu) & O(\mu) \\ O(\mu) & a_{\vartheta\vartheta}(\mu,\vartheta(\mu)) \end{pmatrix}.$$

We have seen that $\mathrm{A}_{\vartheta(\mu)}|E^1_{\vartheta(\mu)}$ is invertible; we have also assumed that ϑ_0 is a simple zero of $\mathcal{M}$, i.e. $\mathcal{M}'(\vartheta_0) \neq 0$; using $a_{\vartheta\vartheta}(\mu,\vartheta(\mu)) = \mu\mathcal{M}'(\vartheta_0) + O(\mu^2)$ one can then show that $\mathrm{d}^2\mathcal{A}_\mu(w_\mu)$ is invertible, for small μ.

Q. E. D.

The size of the splitting. In the classical derivation of the Melnikov function one computes the distance $d(\mu,\vartheta)$ between W^u_μ and W^s_μ in some direction transverse to both invariant manifolds. The Melnikov function arises as the first coefficient of an expansion of $d(\mu,\vartheta)$ in powers of μ: $d(\mu,\vartheta) = \mu\mathcal{M}(\vartheta) + O(\mu^2)$.

One can show that $d(\vartheta,\mu) = \lambda(\vartheta,\mu) + O(\mu^2)$ which gives an approximate interpretation of $\lambda(\mu,\vartheta)$. Instead we will now show that one can give an exact measure of the size of the separation between W^u_μ and W^s_μ in terms of $a(\mu,\vartheta)$.

Homoclinic points lie on the stable manifold of $\mathcal{O}_\mu$ so they can be ordered linearly by their position on this invariant manifold. Their position on the unstable manifold defines another ordering of homoclinic points and these two orderings need not coincide, in general.

Our construction identifies certain homoclinic points with critical points of the function $a(\mu,\cdot)$ so that these particular homoclinics have a third ordering.

For small μ all three of these orderings coincide. That this is so becomes clear if one considers the curve $\gamma_\mu(\vartheta) = (v_{\mu,\vartheta}(0), v'_{\mu,\vartheta}(0))$ with $\vartheta \in \mathbb{R}$. When $\mu = 0$ we have $v_{\mu,\vartheta}(0) = U_\vartheta(0) = U(\vartheta)$, so that $\gamma_\mu(\vartheta) = (U(\vartheta), U'(\vartheta))$ parametrizes the stable and unstable manifolds of $\mathcal{O}_0$. Hence, for small μ the curve parametrized by γ_μ will be C^1 close to both the stable and unstable manifolds of $\mathcal{O}_\mu$. Since the three orderings of critical points are determined by the order in which they occur on the curves γ_μ, W^s_μ and W^u_μ, these orderings coincide for small μ.

2.6. Proposition. *Let $P, Q \in W^u_\mu \cap W^s_\mu$ be two consecutive homoclinic points corresponding to two consecutive critical points $\vartheta_P < \vartheta_Q$ of $a(\mu,\cdot)$. Denote the region bounded by the segments of W^u_μ and W^s_μ that connect P and Q by Ω (some call such a region a "lobe").*

Then the area of the lobe Ω is given by

$$m(\Omega) = |\mathcal{A}_\mu(w_P) - \mathcal{A}_\mu(w_Q)| = |a(\mu,\vartheta_P) - a(\mu,\vartheta_Q)|.$$

This was shown by MacKay, Meiss and Percival[6].

Figure 3 here

[6] See [MMP84] for a discussion of transport by area preserving maps. Their arguments are easily adapted to our setting.

The point of this observation was that it allows one to compute "the flux across a barrier," where the barrier is formed by the stable and unstable manifolds. More precisely, choose a homoclinic point P_μ corresponding to one of the critical points of $a(\mu,\cdot)$; Let $W^s_\mu[P_\mu,\mathcal{O}_\mu]$ and $W^u_\mu[\mathcal{O}_\mu,P_\mu]$ be the segments of the two invariant manifolds between $\mathcal{O}_\mu$ and P_μ; and define Ω to be the bounded region these two curve segments enclose. Then the flux across $\partial\Omega$ is by definition

$$\mathrm{flux}(\Phi_\mu,\Omega) = m\left(\Omega\triangle\Phi_\mu(\Omega)\right).$$

(We write $A\triangle B = (A\cap B^c)\cup(A^c\cap B)$.) Since one can decompose the difference $\Omega\triangle\Phi_\mu(\Omega)$ into a disjoint union of "lobes" one can actually compute the flux in terms of $a(\mu,\cdot)$: One finds that the flux is given by the sum of $|a(\mu,\vartheta_P)-a(\mu,\vartheta_Q)|$ over all consecutive critical points $\vartheta_P<\vartheta_Q$ in one period interval, i.e. the flux is equal to the total variation of $a(\mu,\cdot)$ over $\mathbb{T}$:

$$\mathrm{flux}(\Phi_\mu,\Omega) = \int_{\mathbb{T}} |a_\vartheta(\mu,\vartheta)|\,d\vartheta.$$

For small μ the flux is therefore given by

$$\mathrm{flux}(\Phi_\mu,\Omega) = \mu\int_{\mathbb{T}} |\mathcal{M}(\vartheta)|d\vartheta + O(\mu^2).$$

This completes our discussion of separatrix crossing when the perturbations are small and their period is fixed.

3 Rapidly oscillating perturbations. We turn to equation (1.1), with $\varepsilon>0$ small. In this section we will explain exactly what kind of equations and solutions we consider.

It is well known from the theory of averaging[7] that solutions of (1.1) are approximated by those of the averaged equation,

$$u''(t) = F_0(u(t)), \tag{3.1}$$

where

$$F_0(u) = \int_0^1 F(\tau,u)\,d\tau. \tag{3.2}$$

Indeed, if we define

$$G(t,u) = \int_0^t \left(F(\tau,u)-F_0(u)\right)\,d\tau,$$

then (1.1) can be rewritten as $u''=F_0(u)+G_t(t/\varepsilon,u)$, and hence as

$$-u''(t)+F_0(u)-\varepsilon\left(\frac{dG(t/\varepsilon,u(t))}{dt}+G_u(t/\varepsilon,u(t))u'(t)\right)=0. \tag{3.3}$$

We shall regard (3.3) as the central equation in our discussion. This equation is a small perturbation of the averaged equation (3.1) and it is at least plausible that the deviation of solutions of (1.1) from those of the averaged equation is of order ε, on any bounded time interval; but to make a more precise statement we must first state our hypotheses.

We assume the autonomous term $F_0(u)$ is as in the previous section (i.e. (2.1) holds, and the corresponding potential energy is such that (2.2) again has a unique positive solution U); in addition to this we require F_0 to be an entire function of $u\in\mathbb{C}$.

[7] See [GH, chapter 4].

Concerning $G(t,u)$ we assume it is holomorphic in u, but only bounded measurable in the time variable. In other words, we assume that G may be written as

$$G(t,u) = \sum_{j=0}^{\infty} g_j(t) u^j, \tag{3.4}$$

with $g_j \in L_\infty(\mathbb{T})$ and $\lim_{n\to\infty} \|g_n\|_{L_\infty}^{1/n} = 0$. We could also say that we require $u \mapsto G(\cdot, u)$ to be an entire function with values in $L_\infty(\mathbb{T})$.

Finally, we assume that $t \mapsto G(t,u)$ is right–continuous and vanishes at $t=0$; i.e.

$$\lim_{t\downarrow 0} G(t,u) = 0 \tag{3.5}$$

uniformly in u, on bounded subsets of $\mathbb{C}$.

Since we do not assume that G is differentiable w.r.t. the time variable, $G_t(t/\varepsilon, u(t))$ cannot be defined by simply substituting $u(t)$ in G_t. Hence we must define what it means for a function to be a solution of (1.1).

3.1. Definition. *A solution is an absolutely continuous function $u(t)$ which satisfies (3.3) in the sense of distributions.*

The right–continuity of G allows us to integrate (3.3), which leads to

$$u(t) = u_0 + tv_0 + \int_0^t (t-s)\left\{F_0(u(s)) - \varepsilon G_u(s/\varepsilon, u(s))u'(s)\right\} ds + \\ + \varepsilon \int_0^t G(s/\varepsilon, u(s))\, ds; \tag{3.6}$$

$$u'(t) = v_0 + \int_0^t \left\{F_0(u(s)) - \varepsilon G_u(s/\varepsilon, u(s))u'(s)\right\} ds + \\ + \varepsilon G(s/\varepsilon, u(s)). \tag{3.7}$$

The second identity shows that the derivative u' of any solution is bounded, i.e. solutions are Lipschitz continuous instead of merely absolutely continuous. The same identity also shows that $\lim_{t\downarrow 0} u'(t) = u'(0+)$ exists for any solution u, so that we can speak of the initial values $(u(0), u'(0+))$ of a solution.

By applying Picard–iteration and Gronwall's lemma to the two integral equations one can show the following.

3.2. Lemma. *For any $\varepsilon \ge 0$ and $u_0, v_0 \in \mathbb{C}$ (3.3) has a solution $u(\varepsilon, u_0, v_0; t)$ on a short enough time interval $-T_-(\varepsilon, u_0, v_0) < t < T_+(\varepsilon, u_0, v_0)$.*

The $T_\pm$ are lower semi continuous functions of ε, u_0, v_0, the solution $u(\varepsilon, u_0, v_0; t)$ depends continuously on $\varepsilon \ge 0$, and is holomorphic in u_0, v_0.

Thus, if the solution $u(t)$ of the averaged equation $u'' = F_0(u)$ with $u(0) = u_0, u'(0) = v_0$ exists on the closed interval $-t_- \le t \le t_+$, then for small enough $\varepsilon > 0$ (3.3) will have a solution on the same time interval, and $u(\varepsilon; t) \to u(t)$, $u'(\varepsilon; t) \to u'(t)$ uniformly on $[t_-, t_+]$.

In terms of the Poincaré–maps Φ_ε this may be stated as

$$\lim_{n\to\infty} \left(\Phi_{t/n}\right)^n (u_0, v_0) = \Psi_t(u_0, v_0),$$

where Ψ_t denotes the flow of the averaged system $u' = v, v' = F_0(u)$.

The two examples. If one assumes $G(t,u)$ does not depend on u, and puts $g(t) = G'(t)$ then one obtains the forced Duffing equation (1.2). Hence in terms of the first example we allow periodic forcing terms $g(t)$ which are distributional derivatives of bounded measurable functions with $\int_{\mathbb{T}} g(t)dt = 0$.

To obtain the second example one should choose $G(t,u) = Z(t)F_0(u)$ where

$$Z(t) = \lfloor t \rfloor - t = \max(n \in \mathbb{Z} : n \leq t) - t.$$

In this case our previous remarks show that the solutions of the difference equation (1.4) converge to the solution of the ODE $u'' = F_0(u)$ – this is ofcourse a well known fact from numerical mathematics.

4 Separatrix splitting. In computing the splitting of the separatrices for small ε we try to follow the approach of section 2 as closely as possible. The main result we shall find in this section is:

4.1. Theorem. *Assume that U is analytic in $|\mathrm{Im} t| < \rho_0$. Then for any $\rho < \rho_0$ there exist $R(\rho), \delta(\rho) > 0$ such that the following holds for $\varepsilon[G]_R < \delta$.*

1. *Φ_ε has a hyperbolic fixed point $\mathcal{O}_\varepsilon$ near the origin; $\mathcal{O}_\varepsilon$ depends analytically on ε^2.*

2. *There is a homoclinic point $P_\varepsilon \in W^s_\varepsilon \cap W^u_\varepsilon$ such that*

$$\mathrm{flux}(\Phi_\varepsilon, \Omega_{\mathcal{O}_\varepsilon P_\varepsilon}) \leq C_\rho e^{-2\pi\rho/\varepsilon},$$

where $\Omega_{\mathcal{O}_\varepsilon P_\varepsilon}$ is the domain enclosed by $W^s_\varepsilon[P_\varepsilon, \mathcal{O}_\varepsilon]$ and $W^u_\varepsilon[\mathcal{O}_\varepsilon, P_\varepsilon]$.

In what follows we will use the following seminorms to measure the size of F and G:

$$[G]_R = \mathrm{ess.sup}\left(|G(t,u)| : t \in \mathbb{T}, |u| \leq R\right).$$

Since F and G are analytic in the u variable these seminorms also control the derivatives w.r.t. $u \in \mathbb{C}$ of F and G. Indeed, Cauchy's theorem implies that for almost every $t \in \mathbb{T}$

$$\left|\frac{\partial^k G}{\partial u^k}(t,u)\right| \leq k![G]_{R+1}, \tag{4.1}$$

if $|u| \leq R$.

We shall frequently use this observation in the following form. If we approximate F_0 or G by a Taylor series, then we can estimate the error in terms of the seminorms $[F_0]$ and $[G]$. Indeed,

$$G(t,u+v) = \sum_{k=0}^{n-1} \frac{\partial^k G}{\partial t^k}(t,u)v^k + R_n(u,v),$$

where

$$R_n(u,v) = v^n \int_0^1 \frac{(1-\lambda)^{n-1}}{(n-1)!} \frac{\partial^n G}{\partial u^n}(t, u+\lambda v)\, d\lambda,$$

so that it follows from (4.1) that

$$|R_n(u,v)| \leq [G]_{|u|+|v|+1}|v|^n. \tag{4.2}$$

Step 1–The hyperbolic fixed point. As in section 2 we verify that the hyperbolic fixed point at the origin persists for small $\varepsilon > 0$, when the equation (1.1) is perturbed.

4.2. Lemma. *There is a* $\delta_0 > 0$ *such that (1.1) has an* ε *periodic solution* $p_\varepsilon(t) = p(\varepsilon, t/\varepsilon)$ *for any* ε, G *with* $\varepsilon < \delta_0$ *and* $\varepsilon[G]_2 < \delta_0$. *The solution* $p(\varepsilon, t)$ *is analytic in* ε *and may be written as* $p(\varepsilon, t) = \sum_{j \geq 1} p_j(t)\varepsilon^{2j}$. *The* $W^1_\infty(\mathbb{T})$ *norm of the solution* $p(\varepsilon, \cdot)$ *is estimated by:*

$$\|p(\varepsilon, \cdot)\| \leq C\varepsilon^2[G]_2.$$

In terms of the Poincaré map this means that Φ_ε has a fixed point

$$\mathcal{O}_\varepsilon = (p(\varepsilon, 0), p'(\varepsilon, 0)/\varepsilon),$$

for $0 < \varepsilon < \min(\delta_0, \delta_0/[G]_2)$. We postpone the proof of this lemma until section 6.

Step 2–The modified equation. After substituting $u(t) = v(t) + p(\varepsilon, t/\varepsilon)$ in (1.1) one gets the following equation for v:

$$v'' = \hat{F}(t/\varepsilon, v(t)) \tag{4.3}$$

where $\hat{F}(t, v) = F(t, p(\varepsilon, t) + v) - \varepsilon^{-2}p''(\varepsilon, t)$.

As long as it causes no confusion we will write $p(t)$ instead of $p(\varepsilon, t)$ and we will also suppress the ε dependence of $\hat{F}$.

The averaged equation is $v'' = \hat{F}_0(v)$, where

$$\begin{aligned} \hat{F}_0(v) &= \int_{\mathbb{T}} \{F(t, v + p(t)) - \varepsilon^{-2}p''(t)\}\, dt \\ &= \int_{\mathbb{T}} \{F_0(v + p(t)) - G_u(t, v + p(t))p'(t)\}\, dt. \end{aligned}$$

The average free part of $\hat{F}$ is given by $\hat{G}_t(t, v)$ where

$$\begin{aligned} \hat{G}(t, v) &= \int_0^t \left\{\hat{F}(\tau, v + p(\tau)) - \varepsilon^{-2}p''(\tau)\right\} d\tau - t\hat{F}_0(v) \\ &= \int_0^t \left\{F_0(v + p(\tau)) - \hat{F}_0(v) - G_u(\tau, v + p(\tau))p'(\tau)\right\} d\tau + \\ &\quad + \frac{p'(0) - p'(t)}{\varepsilon^2} + G(\tau, p(\tau) + v). \end{aligned}$$

We also define the modified potentials $\hat{V}_0(v) = \int_0^v \hat{F}_0(w)dw$.

Equation (4.3) is equivalent to (3.3), with F_0 and G replaced by $\hat{F}_0, \hat{G}$. Solutions of (4.3) are exactly the critical points of

$$\mathcal{A}_\varepsilon(v) = \int_{\mathbb{R}} \left\{{}^1/_2 v'(t)^2 - \hat{V}_0(v(t)) - \varepsilon\hat{G}(t/\varepsilon, v(t))v'(t)\right\} dt.$$

The modified nonlinearities $\hat{F}$ and $\hat{G}$ are close to F, G when ε is small. How close they are is measured by the following lemma.

4.3. Lemma. *Let* $\delta_0 > 0$ *be as in lemma 4.1. Then for* $\varepsilon < \delta_0, \varepsilon[G]_2 < \delta_0$ *one has*

$$|\hat{F}_0(v) - F_0(v)| \leq c\,(1 + [G]_{R+2})\,[G]_2\varepsilon^2.$$

$$\left|\hat{G}(t, v) - G(t, v + p(t)) - \frac{p'(0) - p'(t)}{\varepsilon^2}\right| \leq c\,(1 + [G]_{R+2})\,[G]_2\varepsilon^2.$$

$$|\hat{G}(t, v)| \leq c[G]_{R+2}$$

$$|\hat{V}_0(v) - V_0(v)| \leq c\,(1 + [G]_{R+2})\,[G]_2\varepsilon^2$$

whenever $|v| \le R$ *and* $|w| \le 1$. *Here* c *is a constant which only depends on* R *and* $[F_0]_{R+2}$.

The first two inequalities are obtained from the definitions of $\hat{F}_0$ and $\hat{G}$ and our estimate in lemma 4.1, i.e. $|p| + |p'| \le c[G]_2\varepsilon^2$. The third inequality then follows from the second, and one obtains the last one by integrating first estimate.

One can improve these inequalities a little by observing that we have defined $\hat{F}, \hat{G}$ and $\hat{V}$ so that

$$F_0(0) = \hat{F}_0(0) = \hat{G}(t,0) = 0,$$
$$V_0(0) = V_0'(0) = \hat{V}_0(0) = \hat{V}_0'(0) = 0.$$

Since all these functions are analytic in the u variable this means that

$$|G(t,v)| \le c[G]_{R+2} \cdot \left|\frac{v}{R}\right|,$$
$$|\hat{V}_0(v) - V_0(v)| \le c\,(1 + [G]_{R+2})\,[G]_2\varepsilon^2 \cdot \left|\frac{v}{R}\right|^2$$

for $|v| \le R$. This has the following implications for the functional $\mathcal{A}_\varepsilon$.

4.4. Lemma. *Assume again that* $\varepsilon < \delta_0, \varepsilon[G]_2 < \delta_0$. *Then* $\mathcal{A}_\varepsilon$ *is a holomorphic function on* $H^1(\mathbb{R}, \mathbb{C})$, *which satisfies*

$$|\mathcal{A}_\varepsilon(v) - \mathcal{A}_0(v)| \le c[G]_{R+2}\varepsilon,$$

for any $v \in H^1(\mathbb{R}, \mathbb{C})$ *with* $\|v\|_{H^1} \le R$. *Moreover,*

$$\left\|\mathrm{d}^k\mathcal{A}_\varepsilon(v) - \mathrm{d}^k\mathcal{A}_0(v)\right\| \le k!c[G]_{R+2}\varepsilon$$

holds whenever $\|v\|_{H^1} \le R - 1$.

Here and in the following sections we let $H^r = H^r(\mathbb{R}, \mathbb{C})$ stand for the Sobolev space of complex valued functions with r derivatives in L_2.

Proof. For any $v \in H^1(\mathbb{R})$ one has $\|v\|_{L_\infty} \le \|v\|_{H^1}$ so that $\|v\|_{H_1} \le R$ implies $\|v\|_{L_\infty} \le R$.

The first part of this lemma now follows by applying our estimates of $\hat{G}(t,v)$ and $|\hat{V}_0(v) - V_0(v)|$ to

$$\mathcal{A}_\varepsilon(v) - \mathcal{A}_0(v) = \int_{\mathbb{R}} \left\{V_0(v) - \hat{V}_0(v) - \varepsilon G(t/\varepsilon, v)v'\right\} dt.$$

The estimate for the derivatives then follows from Cauchy's formula and the analyticity of $\mathcal{A}_\varepsilon - \mathcal{A}_0$.

Step 3–The curve of critical points. As in section 2 the translates of U form a curve of critical points of $\mathcal{A}_0$. But in the present situation it turns out to be expedient to note that $U(t)$ is an analytic function defined on a neighborhood of the real axis: Since $U(t)$ is analytic in $S_{\rho_0} = \{t \in \mathbb{C} : |\mathrm{Im}t| < \rho_0\}$, every translate $U_\vartheta(t) = U(\vartheta + t)$ with $\vartheta \in S_{\rho_0}$ will also be a critical point of $\mathcal{A}_0$. The curve Σ of critical points is therefore a holomorphic curve in the *complex* Hilbert space $H^1(\mathbb{R}, \mathbb{C})$.

We define $E_\vartheta^{\pm 1}$ to be the subspaces of $H^{\pm 1}$ consisting of all w with $\int_{\mathbb{R}} wU_\vartheta{}' = 0$. The reader should note that for non-real values of ϑ this space is the L_2–orthogonal complement of the complex conjugate of $U_\vartheta{}'$ rather than $U_\vartheta{}'$ itself. With this definition the spaces $E_\vartheta^{\pm 1}$ depend holomorphically on ϑ.

Analogously to lemma 2.2 one now shows

4.5. Lemma. $\mathrm{A}_\vartheta = \mathrm{d}^2\mathcal{A}_0(U_\vartheta) : H^1 \to H^{-1}$ *is Fredholm with index zero.*

$\mathrm{A}_\vartheta | E^1_\vartheta$ *is injective, the range of* A_ϑ *is* E^{-1}_ϑ*, its kernel is spanned by* $U_\vartheta{}'$.

U_ϑ *is a nondegenerate critical point of* $\mathcal{A}|E^1_\vartheta$.

The stability of nondegenerate critical points implies that any small perturbation of $\mathcal{A}_0|E^1_\vartheta$ will have a nondegenerate critical point near U_ϑ. In particular, $\mathcal{A}_\varepsilon|E^1_\vartheta$ will have such a critical point $v_{\vartheta,\varepsilon} \in E^1_\vartheta$ if ε is small enough.

4.6. Lemma. *For any* $\rho < \rho_0$ *there is a* $\delta = \delta(\rho) > 0$ *such that* $\mathcal{A}_\varepsilon|E^1_\vartheta$ *has a unique critical point* $v_{\varepsilon,\vartheta} \in E^1_\vartheta$ *with*

$$\|v_{\varepsilon,\vartheta} - U_\vartheta\|_{H^1} \le c[G]_{R+2}\varepsilon$$

for any $\vartheta \in S_\rho$. *Here* $R = 1 + \sup_{\vartheta \in S_\rho} \|U_\vartheta\|_{H^1}$*, and* c *is a constant that only depends on* ρ *and* $[F_0]_{R+2}$.

The $v_{\varepsilon,\vartheta}$ *are holomorphic in* ϑ*, and* ε *periodic in the sense that* $v_{\varepsilon,\vartheta+\varepsilon}(t) = v_{\vartheta,\varepsilon}(t+\varepsilon)$.

As H^1 *valued function of* ϑ *they are also* C^1 *close to* U_ϑ*, with*

$$\left\| \frac{\partial v_{\varepsilon,\vartheta}}{\partial \vartheta} - U_\vartheta{}' \right\|_{H^1} \le c[G]_{R+2}\varepsilon.$$

We shall obtain $v_{\vartheta,\varepsilon}$ from the Lagrange multiplier equation

$$\mathrm{d}\mathcal{A}_\varepsilon(v) = \lambda U_\vartheta{}', \qquad \langle v, U_\vartheta{}' \rangle = 0 \tag{4.4}$$

in section 7.

Once again the $v_{\varepsilon,\vartheta}$ are not necessarily critical points of $\mathcal{A}_\varepsilon$, but any critical point of $\mathcal{A}_\varepsilon$ which is close to some U_ϑ must be a $v_{\varepsilon,\vartheta}$. At the same time, a $v_{\varepsilon,\vartheta}$ is a critical point i*ff* the corresponding Lagrange multiplier $\lambda(\varepsilon,\vartheta)$, defined by (4.4), vanishes.

Step 5–The function $a(\varepsilon,\vartheta)$. Consider $a(\varepsilon,\vartheta) = \mathcal{A}_\varepsilon(U_\vartheta + w_{\varepsilon,\vartheta})$. Then $a(\varepsilon,\vartheta+\varepsilon) = a(\varepsilon,\vartheta)$ for all $\vartheta \in S_\rho$.

Just as in section 2 the critical points of $a(\varepsilon,\cdot)$ correspond precisely to the homoclinics of the Poincaré-map. Indeed, for $\vartheta \in S_\rho$ and $\varepsilon[G]_{R+2} \le \delta(\rho)$ we have

$$\frac{\partial a}{\partial \vartheta}(\varepsilon,\vartheta) = \lambda(\varepsilon,\vartheta) \left\langle U_\vartheta{}', \frac{\partial v_{\varepsilon,\vartheta}}{\partial \vartheta} \right\rangle = \lambda(\varepsilon,\vartheta)\left(\Gamma + O(\varepsilon[G]_{R+2})\right), \tag{4.5}$$

where $\Gamma = \int_{\mathbb{R}} U'(t+\vartheta)^2 dt > 0$ does not depend on ϑ.

Unlike the case of the Melnikov function we cannot expand $a(\varepsilon,\vartheta)$ in powers of ε. For the moment we shall be satisfied with an upper estimate for the total variation of $a(\varepsilon,\cdot)$.

We begin with a crude estimate:

$$\begin{aligned} |a(\varepsilon,\vartheta) - \mathcal{A}(U_\vartheta)| &\le |\mathcal{A}_0(U_\vartheta + w_{\varepsilon,\vartheta}) - \mathcal{A}_0(U_\vartheta)| + |\mathcal{B}_\varepsilon(U_\vartheta + w_{\varepsilon,\vartheta})| \\ &\le \sup_{\|z\| \le R+2} |\mathcal{A}_0(z)| \cdot \|w_{\varepsilon,\vartheta}\|^2 + c[G]_{R+2}\varepsilon \\ &\le c[G]_{R+2}\varepsilon. \end{aligned}$$

Expand $a(\varepsilon,\vartheta)$ in a Fourier series: if

$$a(\varepsilon,\vartheta) = \sum_n a_n \exp(2\pi i n \vartheta/\varepsilon),$$

then

$$\begin{aligned} |a_n| &\le e^{-2\pi|n|\rho/\varepsilon} \frac{1}{\varepsilon} \int_0^\varepsilon |a(t \pm i\rho)|\, dt \\ &\le c\varepsilon[G]_{R+2} e^{-2\pi|n|\rho/\varepsilon}. \end{aligned}$$

Hence for real valued ϑ we get:

$$|a_\vartheta(\varepsilon,\vartheta)| \le \frac{2\pi}{\varepsilon}\sum_n n|a_n| \le c[G]_{R+2}e^{-2\pi\rho/\varepsilon}.$$

Define the region $\Omega_{\mathcal{O}_\varepsilon P_\varepsilon}$ as at the end of section 2: i.e. let P_ε be a homoclinic point associated with a zero of $a_\vartheta(\varepsilon,\cdot)$. Such a point certainly exists, since $a(\varepsilon,\vartheta)$ is a periodic function of ϑ– simply take a point $\vartheta \in \mathbb{R}$ where $a(\varepsilon,\cdot)$ is maximal, or minimal. Now let $\Omega_{\mathcal{O}_\varepsilon P_\varepsilon}$ be the bounded region enclosed by $W^s_\varepsilon[P_\varepsilon,\mathcal{O}_\varepsilon]$ and $W^u_\varepsilon[\mathcal{O}_\varepsilon,P_\varepsilon]$. Then we find that

$$\begin{aligned}\text{flux}(\Phi_\varepsilon,\Omega) &\le \int_0^\varepsilon |a_\vartheta(\varepsilon,\vartheta)|d\vartheta \\ &\le c[G]_{R+2}\varepsilon e^{-2\pi\rho/\varepsilon}.\end{aligned}$$

5 Holomorphic Contraction Mapping Lemma. In what follows we shall frequently use the contraction mapping lemma to obtain solutions of differential equations. Since we are working with holomorphic maps of Banach spaces the usual contraction mapping lemma admits a stronger formulation. In practice this stronger version will not allow us to prove anything we couldn't do with the usual version, but it does save us a small amount of work: We no longer have to go through the calculations which show that the maps we consider are contractions.

5.1. HCM–lemma. *Let E be a complex Banach space, and let $f : B_r \to B_{\theta r}$ be a holomorphic mapping, where $B_\rho = \{x \in E : \|x\| < \rho\}$.*

If $\theta < {}^1/_2$, then $f|B_{\theta r}$ is a contraction, and hence has a unique fixed point in $B_{\theta r}$.

If F is another complex Banach space, and $f : F \times E \to E$ is holomorphic such that $f(\vartheta_0,\cdot)$ maps B_r into $B_{\theta r}$ for some $\theta < {}^1/_2$, $\vartheta_0 \in F$, then $f(\vartheta,\cdot)$ has a unique fixed point $x(\vartheta) \in B_{\theta r}$ for $\vartheta \in F$ close enough to ϑ_0. This fixed point depends holomorphically on ϑ, and

$$\|\mathrm{d}x(\vartheta_0)\| \le \frac{1-\theta}{1-2\theta}\|\mathrm{d}_\vartheta f(\vartheta_0,x(\vartheta_0))\|.$$

Proof. Use Cauchy's inequality to estimate the derivative of f on $B_{\theta r}$ in terms of its sup norm on B_r:

$$\begin{aligned}\|\mathrm{d}f(x)\| &\le \frac{1}{\rho}\sup\left(\|f(y)\| : \|y-x\| < \rho\right) \\ &\le \frac{\sup\left(\|f(y)\| : y \in B_r\right)}{r-\|x\|} \\ &\le \frac{\theta r}{r-\theta r} = \frac{\theta}{1-\theta} \\ &< 1,\end{aligned}$$

where we chose $\rho = r - \|x\|$, and used $\theta < {}^1/_2$. Thus f is a contraction on $B_{\theta r}$ which must have a fixed point.

By the implicit function theorem the fixed point $x(\vartheta)$ depends smoothly (i.e. holomorphically) on parameters; the estimate for $\mathrm{d}x(\vartheta_0)$ follows directly after differentiating $x(\vartheta) = f(\vartheta, x(\vartheta))$.

Q. E. D.

5.2. HCM–lemma, special case. *If the holomorphic map $f : E \to E$ satisfies*

$$\|f(x)\| \le a + b\|x\|^2,$$

with $ab < 1/16$, then f has a unique fixed point x_0 with $\|x\| \le 2a$.

The special case follows from the HCM–lemma, after observing that $f(B_{4a}) \subset B_{4\theta a}$, where $\theta = (1+16ab)/4 < {}^1/_2$.

6 Proof of lemma 4.1. We define $W^1_\infty(\mathbb{T})$ to be the space of all complex valued Lipschitz continuous functions on $\mathbb{T}$. By Rademacher's theorem $W^1_\infty(\mathbb{T})$ consists of those distributions on $\mathbb{T}$ whose first derivative can be represented by a bounded measurable function. $W^1_\infty(\mathbb{T})$ is a complex Banach space, with norm

$$\|f\|_{W^1_\infty} =_{\text{def}} \|f\|_{L_\infty} + \|f'\|_{L_\infty}.$$

We define $W^{-1}_\infty(\mathbb{T})$ to be the space of all distributions f which may be written as $f = g + h'$, for certain $g, h \in L_\infty(\mathbb{T})$. With the following norm:

$$\|f\|_{W^{-1}_\infty} =_{\text{def}} \inf\left(\|g\|_{L_\infty} + \|h\|_{L_\infty} : f = g + h'\right)$$

$W^{-1}_\infty(\mathbb{T})$ is a complex Banach space. One can identify this space as the dual of $W^1_1(\mathbb{T})$, the space of absolutely continuous functions on $\mathbb{T}$.

One substitutes $u(t) = p(t/\varepsilon)$ in (1.1), or (3.3) and finds the following equation for p:

$$\mathrm{D}^2 p(s) - \varepsilon^2 F(s, p(s)) = 0, \tag{6.1}$$

where the linear operator $\mathrm{D}^2 : W^1_\infty(\mathbb{T}) \to W^{-1}_\infty(\mathbb{T})$ is given by $\mathrm{D}^2 p(s) = p''(s)$. This operator is bounded, Fredholm; its kernel consists of all constant functions, its range is $X^- = \left\{q \in W^{-1}_\infty(\mathbb{T}) : \int_{\mathbb{T}} q = 0\right\}$. When restricted to $X^+ = X^- \cap W^1_\infty(\mathbb{T})$ D^2 is injective. We denote its inverse by $\mathrm{K} : X^- \to X^+$, and extend it to an operator $\mathrm{K} : W^{-1}_\infty(\mathbb{T}) \to X^+ \subset W^1_\infty(\mathbb{T})$ by defining $\mathrm{K}c = 0$ for any constant function c.

Write $p(s)$ as $p(s) = c + q(s)$, where c is constant and $\int_{\mathbb{T}} q(s)ds = 0$. Then $q'' = p''$, so that application of K to (6.1) leads to

$$q = -\varepsilon^2 \mathrm{K}\left\{F_0(c + q(s)) + G_t(s, c + q(s))\right\} =_{\text{def}} \varphi_\varepsilon(c, q). \tag{6.2}$$

Here φ_ε is a holomorphic map from $\mathbb{C} \times X^+$ to X^+. If $|c| \le 1, \|q\| \le 1$ then φ_ε can be estimated by

$$\|\varphi_\varepsilon(c, q)\| \le C\varepsilon^2\left(|c| + \|q\| + [G]_2\right),$$

where $\|\cdot\|$ denotes the $W^1_\infty(\mathbb{T})$ norm. Applying the HCM–lemma, it follows that for small enough ε and any $|c| \le 1$ there is a unique small solution $q_{\varepsilon,c}$ of $q = \varphi_\varepsilon(c, q)$. This solution is a holomorphic X^+–valued function of c and ε, and satisfies

$$\|q\| \le C\varepsilon^2\left(|c| + [G]_2\right). \tag{6.3}$$

The function $c + q_{\varepsilon,c}$ will satisfy (6.1) upto a constant. Requiring this constant to vanish will give us one additional equation, which determines c. To get this equation we integrate (6.1), and write $F_0(u) = F_0'(0)u + F_2(u)u^2$. This leads to

$$c = \frac{-1}{F_0'(0)} \int_{\mathbb{T}} \left\{F_2(p)p^2 - G_u(s, p)q'\right\} ds =_{\text{def}} \psi_\varepsilon(c).$$

Here $p = c + q_{\varepsilon,c}$, and we have replaced $G_t(s, p)$ by $(G(s, p))' - G_u(s, p)p'$. Using (6.3) we get the following estimate for ψ_ε:

$$|\psi_\varepsilon(c)| \le C|c|^2 + C\varepsilon^2[G]_2^2,$$

If $\varepsilon[G]_2$ is small then we can again apply the HCM–lemma and conclude that ψ_ε has a fixed point c_ε with

$$|c_\varepsilon| \le C\varepsilon^2[G]_2^2.$$

We obtain the desired solution by adding c and q: $p(\varepsilon, \cdot) = c_\varepsilon + q_{\varepsilon,c_\varepsilon}$. That $p(\varepsilon, \cdot)$ only contains even powers of ε follows from the uniqueness of the solution for fixed ε, and from the fact that the original equation (6.1) does not change if one changes the sign of ε.

7 Proof of Lemma 4.5. Consider the Euler-Lagrange equation (4.4). In this equation we substitute $v = U_\vartheta + w$, we write $\mathcal{A}_\varepsilon(v) = \mathcal{A}_0(v) + \mathcal{B}_\varepsilon(v)$, and we expand $\mathrm{d}\mathcal{A}_0(v)$ in a Taylor series around U_ϑ:

$$\begin{aligned}\mathrm{d}\mathcal{A}_0(U_\vartheta + w) &= w'' + F_0(U_\vartheta + w) - F_0(U_\vartheta)\\ &= \left\{-\mathrm{D}^2 + F_0'(U_\vartheta(t))\right\} w + R_2(U_\vartheta, w)\\ &= \mathrm{A}_\vartheta w + R_2(U_\vartheta, w)\end{aligned}$$

where R_2 may be estimated by (4.2), i.e. if $|U_\vartheta(t)| \le R$ and $|w| \le 1$, then $|R_2(U_\vartheta(t), w)| \le [F_0]_{R+2}|w|^2$.

Thus $w = v - U_\vartheta$ satisfies

$$\mathrm{A}_\vartheta w + R_2(U_\vartheta, w) + \mathrm{d}\mathcal{B}_\varepsilon(U_\vartheta + w) = \lambda U_\vartheta', \qquad \langle w, U_\vartheta'\rangle = 0. \tag{7.1}$$

Let $\mathrm{T}_\vartheta : H^{-1} \to H^1$ be the pseudo inverse of A_ϑ; i.e.

$$\mathrm{T}_\vartheta|E_\vartheta^{-1} = \left(\mathrm{A}_\vartheta|E_\vartheta^1\right)^{-1}, \quad \text{and} \quad \mathrm{T}_\vartheta U_\vartheta' = 0.$$

The range of T_ϑ is E_ϑ^1. One can compute T_ϑ from the resolvent of A_ϑ via the following formula[8]

$$\mathrm{T}_\vartheta = \frac{-1}{2\pi i}\oint_{|\zeta|=r} (\zeta - \mathrm{A}_\vartheta)^{-1}\frac{d\zeta}{\zeta}.$$

Here $r > 0$ should be chosen small enough, so that $\zeta - \mathrm{A}_\vartheta$ has a bounded inverse for any ζ with $0 < |\zeta| \le r$.

This expression for T_ϑ shows that T_ϑ depends holomorphically on $\vartheta \in S_{\rho_0}$ and that for any $0 < \rho < \rho_0$

$$\sup_{\vartheta\in S_\rho} \|\mathrm{T}_\vartheta\|_{H^{-1}\to H^1} = C_\rho < \infty.$$

Now we apply T_ϑ to both sides of (7.1) to get

$$\begin{aligned} w &= -\mathrm{T}_\vartheta\left\{R_2(U_\vartheta, w) + \mathrm{d}\mathcal{B}_\varepsilon(U_\vartheta + w)\right\}\\ &=_{\text{def}} \varphi(\varepsilon, \vartheta; w).\end{aligned}$$

Clearly $\varphi(\varepsilon, \vartheta; \cdot)$ is a holomorphic map of H^1 to itself. If $\|U_\vartheta\| \le R - 1$ and $\|w\| \le 1$ then we have

$$\begin{aligned}\|\varphi(\varepsilon, \vartheta; w)\| &\le \|\mathrm{T}_\vartheta\|\left\{\|R_2(U_\vartheta, w)\|_{H^{-1}} + \|\mathrm{d}\mathcal{B}_\varepsilon(U_\vartheta + w)\|_{H^{-1}}\right\}\\ &\le C_\rho\left\{[F_0]_{R+2}\|w\|^2 + c[G]_{R+2}\varepsilon\right\}\\ &\le c[G]_{R+2}\varepsilon + c\|w\|^2,\end{aligned}$$

where c only depends on ρ, R and $[F_0]_{R+2}$. By the HCM-lemma there is a $\delta_1 > 0$ such that $\varphi(\varepsilon, \vartheta; \cdot)$ has a unique fixed point $w_{\varepsilon,\vartheta}$ with

$$\|w_{\varepsilon,\vartheta}\| \le c[G]_{R+2}\varepsilon \tag{7.2}$$

as long as $[G]_{R+2}\varepsilon < \delta_1$.

It is clear from the definition of $\varphi(\varepsilon, \vartheta, w)$ that φ is holomorphic in ϑ, with

$$\frac{\partial\varphi}{\partial\vartheta} = -\frac{\partial \mathrm{T}_\vartheta}{\partial\vartheta}\{\cdots\} - \mathrm{T}_\vartheta\left\{\frac{\partial R_2(U_\vartheta, w)}{\partial U_\vartheta}U_\vartheta' + \mathrm{d}^2\mathcal{B}_\varepsilon(U_\vartheta + w)U_\vartheta'\right\}.$$

Thus

$$\left\|\frac{\partial\varphi}{\partial\vartheta}\right\| \le c\left([G]_{R+2}\varepsilon + \|w\|^2\right),$$

and the HCM–lemma implies that $w_{\varepsilon,\vartheta}$ is also holomorphic in ϑ, with $\|\partial_\vartheta w_{\varepsilon,\vartheta}\| \le c[G]_{R+2}\varepsilon$.

[8] See [DS88].

8 The forced Duffing equation. So far we haven't found an explicit formula for $a(\varepsilon, \vartheta)$, for small ε. In this section we show how one can do this for the forced Duffing equation

$$u'' - u + {}^3/_2 u^2 = g(t/\varepsilon), \tag{8.1}$$

provided $g(t) = G'(t)$, and $\|G\|_{L_\infty}$ is small enough:

8.1. Theorem. *If $g(t) = \delta\varepsilon^{10} H'(t)$ where $\|H\| \in L_\infty(\mathbb{T})$ satisfies $\int_\mathbb{T} H = 0$, then*

$$\frac{\partial a}{\partial \vartheta} = \delta\varepsilon^8 e^{-2\pi^2/\varepsilon} \left\{ -32\pi^3 |\hat{H}_1| \sin\left(\frac{2\pi\vartheta}{\varepsilon} - \phi\right) + o(1) \right\}, \qquad (\delta, \varepsilon \to 0)$$

uniformly in $\vartheta \in \mathbb{R}$, where $\hat{H}_1$ is the first Fourier coefficient of $H(t)$, and $\phi = \arg \hat{H}_1$.

If $\hat{H}_1 \neq 0$ then, if ε and δ are small enough, a_ϑ has precisely two nondegenerate zeroes in any period interval, and hence the Poincaré–map $\Phi_{\delta,\varepsilon}$ has two transverse homoclinic orbits.

The small solution. Let $p(\varepsilon, \tau)$ be the small solution, i.e. $p(\varepsilon, \tau + 1) \equiv p(\varepsilon, \tau)$ and $p(t) = p(\varepsilon, t/\varepsilon)$ satisfies (8.1). Since the perturbation $G(t, u) \equiv G(t)$ does not depend on u, all the seminorms $[G]_R = \|G\|_{L_\infty}$ are the same, and p satisfies $\|p\|_{L_\infty} \leq c\|G\|_{L_\infty}\varepsilon^2$, provided $\|G\|_{L_\infty}\varepsilon < \delta_0$. To calculate a_ϑ for small ε, it seems that we need to now the first order term in the expansion of p.

8.2. Lemma. $\|p + \varepsilon^2 \mathrm{K} g\|_{W^1_\infty} \leq C\varepsilon^4 \|g\|_{W^{-1}_\infty}$, *where the operator* K *is as in section 5.*

Since $\int_\mathbb{T} H = 0$, it follows from the way K was defined that $(\mathrm{K}H')' = H$, so this lemma implies that

$$\|p' + \delta\varepsilon^{12} H\|_{L_\infty} \leq C\delta\varepsilon^{14}. \tag{8.2}$$

Proof. As in section 6 we write $p = c + q$ with $\int_\mathbb{T} q = 0$ and c a constant. In view of (6.2),

$$q = -\varepsilon^2 \mathrm{K} \left\{ g + F_0(p) \right\},$$

and the estimate for $\|p\|_{W^1_\infty}$, this implies

$$\|q + \varepsilon^2 \mathrm{K} g\|_{W^1_\infty} \leq C\varepsilon^4 \|g\|_{W^{-1}_\infty}.$$

For c we have

$$0 = \int_\mathbb{T} F_0(c + q) dt = F_0(c) + \int_\mathbb{T} \left(F_0(c + q) - F_0(c) - F_0'(c) q \right) dt,$$

which implies $|F_0(c)| \leq C\|q\|^2$, and hence $|c| \leq C\varepsilon^4 \|g\|^2_{W^{-1}_\infty}$. Adding our estimates for c and q yields the desired inequality.

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The modified equation. Put $u = v + p$. Then u corresponds to a homoclinic orbit *iff*

$$v'' - v + {}^3/_2 v^2 + 3p(\varepsilon, t/\varepsilon) v = 0, \qquad v(\pm\infty) = 0. \tag{8.3}$$

The solutions of (8.3) are precisely the critical points of $\mathcal{A}_\varepsilon : H^1 \to \mathbb{C}$, where $\mathcal{A}_\varepsilon = \mathcal{A}_0 + \mathcal{B}_\varepsilon$, and

$$\mathcal{A}_0(v) = \int_\mathbb{R} \left({}^1/_2 v'^2 + {}^1/_2 v^2 - {}^1/_2 v^3 \right) dt,$$

$$\mathcal{B}_\varepsilon(v) = -\int_\mathbb{R} {}^3/_2 p(\varepsilon, t/\varepsilon) v(t)^2 dt.$$

The homoclinic orbit of the unperturbed equation is parametrized by $(U(t), U'(t))$, where

$$U(t) = \operatorname{sech}^2 \frac{t}{2} = -\sum_{k\in\mathbb{Z}} \frac{4}{(t-(2k+1)\pi i)^2}. \tag{8.4}$$

We consider the Euler–Lagrange equations $\mathrm{d}\mathcal{A}_\varepsilon(v) = \lambda U_\vartheta{}'$, which upon substitution of $v = U_\vartheta + w$ may be written as

$$w'' - (1-3U_\vartheta)\, w + {}^3\!/_2 w^2 + 3pU_\vartheta + 3pw = \lambda U_\vartheta{}'. \tag{8.5}$$

If T_ϑ again denotes the pseudo inverse of the operator

$$\mathrm{A}_\vartheta = -\mathrm{D}^2 + F_0'(U_\vartheta) = -\mathrm{D}^2 + (1-3U_\vartheta)\,,$$

then (8.5) is equivalent with

$$w = \mathrm{T}_\vartheta \left\{ 3pU_\vartheta + 3pw + {}^3\!/_2 w^2 \right\}. \tag{8.6}$$

To analyze this equation we use our complete knowledge of the operator A_ϑ to first give an asymptotic description for T_ϑ as $\mathrm{Im}\vartheta \uparrow \pi$, and then use this asymptotics to compute $\lambda(\varepsilon, \vartheta)$ for $\mathrm{Im}\vartheta$ close to π.

The resolvent of A_ϑ. For $\omega \in \mathbb{C} - [1,\infty)$ we denote by $\sqrt{1-\omega}$ the square root with positive real part. If $\omega \in \mathbb{C} - [1,\infty)$ then the equation

$$-y'' + F_0'(U(t))y = \omega y \tag{8.7}$$

has two solutions $y_\pm(\omega, t)$, with

$$\begin{aligned} y_+(\omega,t) &= e^{-t\sqrt{1-\omega}}(1+o(1)), \qquad (t\to\infty),\\ y_-(\omega,t) &= e^{t\sqrt{1-\omega}}(1+o(1)), \qquad (t\to-\infty). \end{aligned}$$

Both solutions are holomorphic functions of $t \in \mathbb{C}$, with $|\mathrm{Im}t| < \pi$; they also depend analytically on ω. Their Wronskian

$$W(\omega) = \begin{vmatrix} y_-(\omega,t) & y_+(\omega,t) \\ y_-'(\omega,t) & y_+'(\omega,t) \end{vmatrix}$$

is therefore a holomorphic function on $\mathbb{C} - [1,\infty)$; $\mathrm{A}_0 - \omega$ is invertible *iff* $W(\omega) \neq 0$. E.g. we know that A_0 itself is not invertible and, indeed, $y_\pm(0,t) = U'(t)$.

It is well known[9] that if $W(\omega) \neq 0$, then the inverse of $\mathrm{A}_0 - \omega$ is an integral operator, with kernel

$$K(\omega; t, s) = \frac{y_+(\omega,t)y_-(\omega,s)}{W(\omega)} \tag{8.8}$$

if $t \geq s$; if $s > t$ then $K(\omega; t, s) = K(\omega; s, t)$.

As $y_\pm(\omega, t+\vartheta)$ are solutions of

$$-y'' + F_0'(U_\vartheta(t))y = \omega y,$$

the next lemma follows in the same way.

[9] E.g. see chapter XIII of [DS88].

8.3. Lemma. *If $W(\omega) \neq 0$ then $\mathrm{A}_\vartheta - \omega : H^1 \to H^{-1}$ is invertible for $\vartheta \in S_\pi$. Its inverse is a bounded integral operator on $L_\infty(\mathbb{R}, \mathbb{C})$ with kernel $K(\omega; t+\vartheta, s+\vartheta)$.*

One can verify that the operator with kernel $K(\omega; t+\vartheta, s+\vartheta)$ is bounded on L_∞ by estimating its norm as follows:

$$\|(\mathrm{A}_\vartheta - \omega)^{-1}\|_{L_\infty \to L_\infty} \leq \text{ess. sup}_{t \in \mathbb{R}} \int_{\mathbb{R}} |K(\omega; t+\vartheta, s+\vartheta)| ds.$$

To analyze the kernel K as $\mathrm{Im} t, \mathrm{Im} s$ get close to π we look at the singular points of the differential equation which defines $y_\pm$. By substituting the expansion (8.4) in (8.7) we find that the points $t = (2k+1)\pi i$ are regular singular points for (8.7); a short computation shows that the charcteristic exponents at the singular points are -3 and $+4$. It follows that the solutions $y_\pm$ of (8.7) satisfy

$$|y_\pm(\omega, t)| \leq C_\omega |t^2 + \pi^2|^{-3}$$

near the two singular points $\pm\pi i$.

Combining this with the exponential growth of $y_\pm(\omega, t)$ at $t = \mp\infty$ we get for all $t \in S_\pi$

$$|y_\pm(\omega, t)| \leq C_\omega \left(1 + |t^2 + \pi^2|^{-3}\right) e^{\mp\kappa \mathrm{Re} t},$$

and thus

$$|K(\omega; t, s)| \leq \frac{C_\omega}{|W(\omega)|} \left(1 + |t^2 + \pi^2|^{-3}\right) \left(1 + |s^2 + \pi^2|^{-3}\right) e^{-\kappa|t-s|},$$

where $\kappa = \mathrm{Re}\sqrt{1-\omega}$.

The kernel of the pseudo inverse T_ϑ. Recall that T_ϑ is obtained by integrating the resolvent around a small circle centered at the origin. Hence T_ϑ is also an integral operator with kernel $T(\vartheta + t, \vartheta + s)$ where

$$T(t, s) = \frac{-1}{2\pi i} \oint_{|\omega|=r} K(\omega; t, s) \frac{d\omega}{\omega},$$

and $r > 0$ should be chosen sufficiently small. Our estimate for K implies

$$|T(t, s)| \leq C_r \left(1 + |t^2 + \pi^2|^{-3}\right) \left(1 + |s^2 + \pi^2|^{-3}\right) e^{-\sqrt{1-r}|t-s|} \tag{8.9}$$

for all $s, t \in S_\pi$.

8.4. Lemma. *For any σ with $0 < \mathrm{Im}\sigma < 2\pi/\varepsilon$*

$$\|\mathrm{T}_{\pi i - \varepsilon\sigma}\|_{L_\infty \to L_\infty} \leq \frac{C(\sigma)}{\varepsilon^5}.$$

Proof. It follows from (8.9) that the kernel of $\mathrm{T}_{\pi i - \varepsilon\sigma}$ is dominated by

$$|T(\pi i + t - \varepsilon\sigma, \pi i + s - \varepsilon\sigma)| \leq \left(1 + |t - \varepsilon\sigma|^{-3}\right) \left(1 + |s - \varepsilon\sigma|^{-3}\right) e^{-\sqrt{1-r}|t-s|}.$$

Hence

$$\int_{\mathbb{R}} |T(\pi i + t - \varepsilon\sigma, \pi i + s - \varepsilon\sigma)| ds \leq \frac{C(\sigma)}{\varepsilon^2} \left(1 + |t - \varepsilon\sigma|^{-3}\right) \leq \frac{C}{\varepsilon^5},$$

from which the lemma follows.

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The fixed point equation with $\mathrm{Im}\vartheta \approx \pi$. We return to (8.6): let

$$\varphi_{\varepsilon,\vartheta}(w) = \mathrm{T}_\vartheta \left\{3pU_\vartheta + 3pw + {}^3/_2 w^2\right\}.$$

Then $\varphi_{\varepsilon,\vartheta} : L_\infty \to L_\infty$ is a holomorphic map.

For $\vartheta = \pi i - \varepsilon\sigma$, $\mathrm{Im}\sigma > 0$ we have:

$$\begin{aligned}\|\varphi_{\varepsilon,\vartheta}\| &\leq \frac{C}{\varepsilon^5}\left\{3\|p\|\left(\|U_\vartheta\| + \|w\|\right) + {}^3/_2\|w\|^2\right\} \\ &\leq \frac{C}{\varepsilon^5}\left\{\|p\|\cdot\|U_\vartheta\| + \|p\|^2 + \|w\|^2\right\} \\ &\leq \frac{C}{\varepsilon^5}\left\{\|g\|_{W^{-1}_\infty} + \varepsilon^4\|g\|^2_{W^{-1}_\infty} + \|w\|^2\right\}.\end{aligned}$$

where we have used that $|U_\vartheta| \leq C(\sigma)\varepsilon^{-2}$, $\|\cdot\|$ denotes the L_∞ norm, and C is a constant depending on σ.

By the HCM–lemma $\varphi_{\varepsilon,\vartheta}$ will have a unique small fixed point $w_{\varepsilon,\vartheta}$ provided $C\varepsilon^{-10}(\|g\| + \varepsilon^4\|g\|^2) < {}^1/_2$, i.e. if

$$\|g\|_{W^{-1}_\infty} \leq C\delta\varepsilon^{10}$$

for some small enough $\delta > 0$. This is why we assumed that $g = \delta\varepsilon^{10}H'$, with $H \in L_\infty$ and $\delta > 0$ sufficiently small.

The fixed point w satisfies $\|w_{\varepsilon,\vartheta}\| \leq C\delta\varepsilon^5$.

The functions $\lambda(\varepsilon,\vartheta)$ *and* $a_\vartheta(\varepsilon,\vartheta)$. Multiplying (8.5) with $U_\vartheta{}'$ and integrating results in

$$\lambda(\varepsilon,\vartheta) = \frac{1}{\Gamma}\int_{\mathbb{R}} U_\vartheta{}'\left\{3pU_\vartheta + 3pw + {}^3/_2 w^2\right\} dt. \tag{8.10}$$

Since $\|p\| \leq C\varepsilon^2\|g\|_{W^{-1}_\infty} \leq C\delta\varepsilon^{12}$, we have

$$\begin{aligned}\int_{\mathbb{R}} U_\vartheta{}'\left\{3pw + {}^3/_2 w^2\right\} dt &\leq C\int_{\mathbb{R}} |U_\vartheta{}'|(|p| + |w|)|w| dt \\ &\leq C\delta^2\left(\varepsilon^{17} + \varepsilon^{10}\right)\int_{\mathbb{R}} |U_\vartheta{}'| dt \\ &\leq C\delta^2\varepsilon^8.\end{aligned}$$

After integarting by parts in the first term of (8.10) we therefore get

$$\lambda(\varepsilon,\vartheta) = -\frac{3}{2\Gamma}\int_{\mathbb{R}} p'(\varepsilon,\tau)U(\vartheta + \varepsilon\tau)^2 d\tau + O\left(\delta^2\varepsilon^8\right). \tag{8.11}$$

By expanding $U(t)$ in a Laurent series around $t = \pi i$, and using the exponential decay of $U(t), t \in S_\pi$ as $\mathrm{Re}t \to \pm\infty$ one obtains

$$\left|U(\vartheta + \varepsilon\tau)^2 - \frac{16}{\varepsilon^4(\sigma - \tau)^4}\right| \leq \frac{C}{\varepsilon^2|\sigma - \tau|^2}, \qquad \forall\tau \in \mathbb{R}.$$

Combine this with lemma 8.2, and substitute in (8.11). The result is

$$\lambda(\varepsilon,\vartheta) = \frac{-24\delta\varepsilon^8}{\Gamma}\int_{\mathbb{R}} \frac{H(\tau)}{(\sigma - \tau)^4} d\tau + O\left(\delta\varepsilon^{10} + \delta^2\varepsilon^8\right)$$

From this we shall compute the Fourier expansion of $\lambda(\varepsilon,\vartheta)$. Put

$$\lambda(\varepsilon,\vartheta) = \sum_{n\in\mathbb{Z}} \lambda_n(\varepsilon)e^{2\pi i n\vartheta/\varepsilon}; \qquad H(\tau) = \sum_{n\in\mathbb{Z}} \hat{H}_n e^{2\pi i n\tau}.$$

Then contour integration shows

$$\int_{\mathbb{R}} \frac{H(\tau)}{(\sigma-\tau)^4} d\tau = -{}^4/{}_3\pi^3 i \sum_1^\infty n^3 \hat{H}_n e^{2\pi i \sigma}.$$

For $n \geq 1$ we then have

$$\begin{aligned}
\lambda_{-n} &= \frac{1}{\varepsilon}\int_0^\varepsilon e^{2\pi i n\vartheta/\varepsilon}\lambda(\varepsilon,\vartheta)d\vartheta \\
&= -e^{-2\pi^2 n/\varepsilon}\int_i^{i+1}\lambda(\varepsilon,\pi i-\varepsilon\sigma)d\sigma \\
&= -\frac{32\pi^3 i}{\Gamma}e^{-2\pi^2 n/\varepsilon}\delta\varepsilon^8\left(n^3\hat{H}_n + O\left(\delta^2+\delta\varepsilon^2\right)\right),
\end{aligned}$$

where we have substituted $\vartheta = \pi i - \varepsilon\sigma$, and we have used the analyticity of $\lambda(\varepsilon,\cdot)$ to change the path of integration.

For $n > 0$ we have $\lambda_n(\varepsilon) = \overline{\lambda_{-n}(\varepsilon)}$, since λ is a real valued function for real ϑ. The terms in the Fourier series of λ with $n = 0, \pm 1$ dominate those with $|n| \geq 2$, so, writing $\hat{H}_1 = |\hat{H}_1|e^{i\phi}$, we get:

$$\lambda(\varepsilon,\vartheta) = \lambda_0(\varepsilon) - \frac{32\pi^3}{\Gamma}\delta\varepsilon^8 e^{-2\pi^2/\varepsilon}\left\{|\hat{H}_1|\sin\left(\frac{2\pi i}{\varepsilon}-\phi\right) + O\left(\delta+\varepsilon^2\right)\right\}. \tag{8.12}$$

Finally, using

$$a_\vartheta = (\Gamma + o(1))\lambda(\varepsilon,\vartheta), \tag{8.13}$$

it follows from $\int_0^\varepsilon a_\vartheta(\varepsilon,\vartheta)d\vartheta = 0$ that $\lambda_0 = o\left(\delta\varepsilon^8 e^{-2\pi^2/\varepsilon}\right)$; combining (8.13) and (8.12) one then finds the desired expression for a_ϑ.

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