The Zoo of Curve Shortening Solitons in $\mathbb{R}^n$

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A family of space curves $X = X(t, \xi)$, evolves by Curve Shortening if

$$X_t = X_{ss} + \lambda X_s,$$

for some function $\lambda = \lambda(t, \xi)$. 
A family of space curves $X = X(t, \zeta)$, evolves by Curve Shortening if

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for some function $\lambda = \lambda(t, \zeta)$.

Equivalently, $X$ evolves by CS if

$$X_t^\perp = X_{ss},$$

where $X_t = X_t^\perp + \lambda X_s$ is the decomposition of $X_t$ into normal and tangential components.
Curve Shortening is invariant under a large number of transformations of space-time, namely

1. **Translations** $(t, X) \rightarrow (t + a, X + a)$, $a \in \mathbb{R}^n$.
2. **Rotations** $(t, X) \rightarrow (t, R X)$, $R \in SO(n)$.
3. **Parabolic Dilation** $(t, X) \rightarrow (s^2 t, s X)$, $s \in \mathbb{R}^+$. 

The full symmetry group $G$ consists of combinations of these transformations, $g(t, x) = (s^2 t + t, s R x + a)$.
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Self similar evolutions

Definition: *Solitons* are solutions of Curve Shortening that are invariant under a one parameter subgroup of symmetries $\{g_\theta : \theta \in \mathbb{R}\} \subset \mathcal{G}$.
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for some fixed space curve \( C \) that satisfies
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for some real valued function \( \lambda \), where
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- \( v \) is the translation velocity
Self similar solutions without rotation

\[ A = 0, \alpha \in \{0, \pm \frac{1}{2}\}, \mathbf{v} \in \mathbb{R}^n \]

- Dilation
Self similar solutions \textit{without} rotation

$A = 0, \alpha \in \{0, \pm \frac{1}{2}\}, v \in \mathbb{R}^n$

- Dilation
  
  **Expanders** $X(t) = \sqrt{t} C, t > 0$

  \textit{Brakke's 1979 wedge}
Self similar solutions \textit{without} rotation

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\begin{itemize}
  \item Dilation
  \begin{align*}
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  \text{Shrinkers} \quad & X(t) = \sqrt{-t} \mathbf{C}, \ t < 0
  \end{align*}

  \text{Brakke’s 1979 wedge} \quad \text{Circles, Abresch–Langer curves}
\end{itemize}
Self similar solutions *without* rotation

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  **Shrinkers** \( X(t) = \sqrt{-t} \mathbf{C}, t < 0 \)

- **Translation:** \( X(t) = \mathbf{C} + tv, \)

- **Shrinkers**
  
  **Abresch-Langer**

- **Expanding wedge**

- **Circle**

- **Translating soliton**

- **Brakke’s 1979 wedge**

- **Circles, Abresch–Langer curves**

- **The Grim Reaper**
What other solitons are there?
Non rotating solitons are planar
(the case $A = 0$ is easy)

**Theorem.** If $C$ is a solution of the non rotating soliton equation

$$C_{ss} = a C + \lambda(s) C_s + v$$

then there is a two dimensional affine subspace of $\mathbb{R}^n$ that contains the curve $C$. 
Theorem. If $\mathbf{C}$ is a solution of the non rotating soliton equation

$$\mathbf{C}_{ss} = \alpha \mathbf{C} + \lambda(s) \mathbf{C}_s + \mathbf{v}$$

then there is a two dimensional affine subspace of $\mathbb{R}^n$ that contains the curve $\mathbf{C}$.

Proof (when $\mathbf{v} = 0$): note that $\alpha$ and $\lambda(s)$ are scalars. If $\phi, \psi$ are the two solutions of the ODE $y'''(s) = \alpha y(s) + \lambda(s)y'(s)$ with

$$\phi(0) = 1, \phi'(0) = 0, \text{ and } \psi(0) = 0, \psi'(0) = 1$$

then

$$\mathbf{C}(s) = \phi(s)\mathbf{C}(0) + \psi(s)\mathbf{C}_s(0).$$
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**Theorem.** If $C$ is a solution of the non rotating soliton equation

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From now on: $A \neq 0$!
Given $A = -A^T \in \mathfrak{so}(n, \mathbb{R})$, and $e^{tA} \in SO(n, \mathbb{R})$ we have the following kinds of solitons:
Self similar solutions with rotation
what kind of rotating solitons can there be?

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- **Pure Rotation:** $X(t) = e^{tA}C$

(for any $z > 0$: $z^A \overset{\text{def}}{=} e^{(\log z)A}$)
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- **Rotation&translation**: $X(t) = e^{tA} C + tv, \ v \in \ker A$
- **Rotation&dilation**:
  - \textit{Expanding} $(t > 0)$ $X(t) = \sqrt{t} t^A C$
  - \textit{Shrinking} $(t < 0)$ $X(t) = \frac{1}{\sqrt{-t}} (-t)^A C$

In both cases, $X(t) = (\pm t)^{\pm \frac{1}{2}} + A C,$

(for any $z > 0$: $z^A \overset{\text{def}}{=} e^{(\log z)A}$)
The soliton equation

\[ C_{ss} = (\alpha + A)C + v + \lambda(s)C_s \]

can be rewritten as a first order system, \( T = C_s \)

\[ C_s = T, \quad T_s = \mathfrak{p}_T[(\alpha + A)C + v] \]

where

\[ \mathfrak{p}_a(b) \overset{\text{def}}{=} b - \langle a,b \rangle a \]

is the orthogonal projection along the unit vector \( a \).

The phase space for this system is

\[ \{(C, T) : \|T\| = 1\} = \mathbb{R}^n \times S^{n-1}. \]
Is the soliton flow a geodesic flow?

The soliton flow

\[ \mathbf{C}_s = T, \quad \mathbf{T}_s = p_T[(\alpha + A)\mathbf{C} + \mathbf{v}] \]

defines a flow on \( \mathbb{R}^n \times S^{n-1} \), so \textit{through each point} \( \mathbf{C} \in \mathbb{R}^n \) \textit{and in each direction} \( \mathbf{T} \in T_{\mathbf{C}}\mathbb{R}^n \) there is a unique soliton.
Is the soliton flow a geodesic flow?

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**Theorem.** When $A = 0$ the soliton flow is the geodesic flow for the metric

$$(ds)^2 = e^{\frac{\alpha}{2}} \|C\|^2 \|dC\|^2$$
on $\mathbb{R}^n$. 

IMA talk

the soliton zoo
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Huisken’s monotonicity formula:

$$\frac{d}{dt} \frac{1}{\sqrt{T-t}} \int_{C(t)} e^{-||x||^2/4(T-t)} ds \leq 0$$
Some 3D pictures

In the next series of pictures we have

$$\alpha \in \mathbb{R} \ldots$$

$$\begin{cases} 
\alpha > 0 & \text{expanding} \\
\alpha < 0 & \text{contracting} \\
\alpha = 0 & \text{no dilation}
\end{cases}$$

$$v = \begin{pmatrix} 
0 \\
0 \\
v_z
\end{pmatrix}, \quad v_z \geq 0,$$

$$A = \begin{pmatrix} 
0 & -\omega & 0 \\
\omega & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \omega > 0$$
Wagon wheels
Translating and rotating
IMA talk
the soliton zoo
From here on assume that $v = 0$ and $\alpha \neq 0$

i.e. only look at
rotating shrinkers and expanders
The ends of rotating expanders

\[ C_{ss} = (\alpha + A)C + \lambda C_s \]

Assume \( \alpha > 0 \) (expanding solitons) and \( A \neq 0 \) (rotating solitons).

- The distance to the origin \( ||C|| \) attains a unique minimum on the soliton,
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- The distance to the origin \( ||C|| \) attains a unique minimum on the soliton,
- \( ||C|| \to \infty \) along both ends of the soliton
- As \( s \to \pm \infty \), we have the asymptotic expansion

\[ C(s) \sim e^{\theta(s)(\alpha + A)} \Gamma_{\pm}, \quad \theta(s) \to \infty \]

for certain constant vectors \( \Gamma_{\pm} \) that depend on the soliton.
Next, rotating *shrinkers*...

\[ \alpha < 0, \quad A = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
Rotating and Shrinking Solitons

various rotation rates
Why should there be a Lyapunov function?

Gradient flow of $\int L(u, u_x) \, dx$ \hspace{1cm} \begin{align*} u_t &= \text{Euler Lagrange} \quad u_t = \frac{\partial L_{u_x}}{\partial x} - L_u. \end{align*}$

E.g.

gradient flow of $\int \left\{ \frac{1}{2} u_x^2 - F(u) \right\} \, dx$ is $u_t = u_{xx} + F'(u).$
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Gradient flow of \( \int L(u, u_x)dx \)
\[
\begin{align*}
\text{Euler} & \quad \text{Lagrange} \\
\text{\Rightarrow} & \\
\text{u}_t & = \frac{dL_{u_x}}{dx} - L_u.
\end{align*}
\]

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\text{gradient flow of } \int \left\{ \frac{1}{2} u_x^2 - F(u) \right\} dx \text{ is } u_t = u_{xx} + F'(u).
\]

Traveling waves: \( u(t, x) = u(x - ct), u_t = cu_x \)

are solutions of the \textit{traveling wave ODE}

\[
cu_x = \frac{dL_{u_x}}{dx} - L_u. \quad \text{e.g. } cu_x = u_{xx} + F'(u).
\]
Why should there be a Lyapunov function?

Travelling wave ODE: \( cu_x = \frac{dL_{ux}}{dx} - L_u \)

The Hamiltonian \( H = u_x L_{ux} - L \) is a Lyapunov function for the traveling wave ODE:

\[
\frac{d}{dx} \left( u_x L_{ux} - L \right) = u_{xx} L_{ux} + u_x \frac{dL_{ux}}{dx} - L_u u_x - L_{ux} u_{xx} \\
= u_x \left\{ \frac{dL_{ux}}{dx} - L_u \right\} \\
= c \left( u_x \right)^2.
\]
The almost Lyapunov function
\(\alpha < 0, v = 0\).

If we assume \(\alpha < 0\), then the soliton flow on the unit tangent bundle,

\[ C_s = T, \quad T_s = p_T[(\alpha + A)C] \]

has an almost Lyapunov function, namely

\[ V(C, T) = \langle T, AC \rangle e^{\frac{\alpha}{2}} \| C \|_2^2. \]

One has

\[ \frac{dV}{ds} = e^{\frac{\alpha}{2}} \| C \|_2^2 \| p_T(AC) \|_2^2. \]

In particular, \(\frac{dV}{ds} \geq 0\) with equality only if \(AC\) is a multiple of \(T = C_s\).

**Critical orbits:** orbits on which \(V\) is constant.
The ends of rotating shrinkers

\[ V(C, T) = \langle T, AC \rangle e^{\alpha \|C\|^2 / 2} \] is an almost Lyapunov function

Let \( C \) be a shrinking rotating soliton. If the end \( \{C(s) : s \geq 0\} \) is bounded, then its \( \omega \)-limit set is a union of critical orbits. The only bounded critical orbit is the circle. So, \textit{bounded ends of a soliton converge to the circle.}

Some rotating shrinkers have unbounded ends (e.g. the \( z \)-axis).
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Some rotating shrinkers have unbounded ends (e.g. the \( z \)-axis).

**Theorem about unbounded ends.** If \( \{C(s) : s \geq 0\} \) is an unbounded end of a rotating shrinking soliton, then either the end is asymptotic to a logarithmic spiral,

\[
(\exists \Gamma \in \mathbb{R}^n) \quad C(s) \sim e^{-\theta(s)(\alpha + A)} \Gamma, \quad \theta(s) \to \infty
\]

or else the \( \omega \)-limit set of \( C \) is contained in the \( z \)-axis.
Could this happen?
This happens...

C = [0, 0.0131, 60]  T = [0 1 0]

r = 0.013053
Linearize near the $z$-axis

\[
\frac{f_{zz}}{1 + |f_z|^2} - zf_z + (1 - i\omega)f = 0.
\]

\[
f_{zz} - zf_z + (1 - i\omega)f = 0.
\]
Linearize near the $z$-axis

\[ f_{zz} - zf_z + (1 - i\omega)f = 0 \implies f(z) = aW(z) + bW(-z), \]

Asymptotics:

\[ W(z) \sim z^{1-i\omega}, \quad W(-z) \sim Kz^{-2+i\omega}e^{z^2/2} \]
Linearize near the $z$-axis

$$f_{zz} - z f_z + (1 - i\omega)f = 0 \implies f(z) = aW(z) + bW(-z),$$

Asymptotics:

$$W(z) \sim z^{1-i\omega}, \quad W(-z) \sim K z^{-2+i\omega} e^{z^2/2}$$

$$|W(z)| \sim z, \quad |W(-z)| \sim e^{z^2/2} + \cdots$$
Up and down the $z$-axis
\[ f(z) = aW(z) + bW(-z), \quad W(z) \sim z^{1-i\omega}, \quad W(-z) \sim Kz^{-2+i\omega}e^{z^2/2} \]

Matching the asymptotics leads to the following prediction of the growth of the amplitudes $M_k$ of the oscillation:

\[ M_{k+1} = e^{\frac{1}{4}M_k^2} + O(\log M_k) \]