Interior Gradient Blow-up
in a Semilinear Parabolic Equation

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Abstract. We present a one dimensional semilinear parabolic equation for which the spatial derivative of solutions becomes unbounded in finite time while the solutions themselves remain bounded. In our example the derivative blows up in the interior of the space interval rather than at the boundary, as in earlier examples. In the case of monotone solutions we show that gradient blow-up occurs at a single point, and we study the shape of the singularity. Our argument for gradient blow-up also provides a pair of "naive viscosity sub- and super-solutions" which violate the comparison principle.

1. Introduction

Consider bounded solutions of the following semilinear parabolic PDE

$$\frac{\partial u}{\partial t} = u_{xx} + G(u, u_x), \qquad (|x| < 1, 0 < t < T) \tag{1.1}$$

$$u(x,0) = \varphi(x), \qquad u(\pm 1, t) = A_\pm. \tag{1.2}$$

It is well known [LSU] that this initial value problem has a unique classical solution for small $T > 0$, provided $G \in C^1(\mathbf{R}^2), \varphi \in C^1([-1,1])$ and provided the initial data φ is compatible with the boundary data, $\varphi(\pm 1) = A_\pm$. It is also well known [LSU], [L] that the derivative of any bounded solution is also uniformly bounded if

$$|G(u,p)| \le C(u)(1+p^2)h(|p|), \qquad h \ge 1, \qquad \int_1^\infty \frac{\mathrm{d}\tau}{\tau h(\tau)} = \infty. \tag{1.3}$$

Similar statements are true for quasilinear equations in one and also in more space dimensions.

Examples showing that condition (1.3) is optimal for gradient bounds are known [L], [FL] (and references in [FL]) but in these examples the derivative always blows up on the boundary of the interval.

For quasilinear degenerate parabolic equations in one space dimension interior derivative blow-up is known to occur [B1]–[B3], [G].

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In this note we present a class of semilinear equations with bounded solutions whose derivative blows up in finite time in the interior of the interval $(-1, 1)$. It turns out that for the equations we consider a naive definition of viscosity solutions does not give a good theory of generalized solutions: we find that the equations have discontinuous viscosity sub- and supersolutions that do not obey the maximum principle.

The equations in question are

$$u_t = u_{xx} + f(u)|u_x|^{m-1}u_x, \tag{1.4}$$

with $f \in C^1(\mathbf{R})$, $m > 2$ a constant, and with the initial-boundary conditions (1.2). Before considering gradient blow-up we first note that, even for (1.4, 2), blow-up cannot occur just anywhere.

1.1. Gradient estimate. *Let $u(x,t)$ be a classical solution of* (1.4) *with $\alpha \le u \le \beta$ on*

$$Q = Q_R(x_0, t_0) = (x_0 - R, x_0 + R) \times (t_0 - R^2, t_0).$$

If $f \in C^1([\alpha, \beta])$, and if

$$\delta = \inf_{\alpha \le s \le \beta} |f(s)| > 0 \tag{1.5}$$

then there is a constant $k > 0$ such that

$$|u_x| \le k \quad \text{on } Q_{R/2}. \tag{1.6}$$

The constant k depends on m, R, $\beta - \alpha$, δ, and $\sup_{[\alpha,\beta]} |f'|$.

It follows directly from this estimate that solutions of (1.4) can only become singular at points where $f(u)$ changes sign. More precisely, if $x_0 \in (-1, 1)$ is a blow-up point, which means that there are $x_n \to x_0$ and $t_n \uparrow T$ for which $|u_x(x_n, t_n)| \to \infty$, then there must also exist $x'_n \to x_0$ and $t'_n \uparrow T$ with $f(u(x'_n, t'_n)) \to 0$.

To simplify matters we shall assume in this paper that f has only one sign change. After adding a suitable constant to u and possibly passing to $v = -u$ instead of u we may assume

$$uf(u) > 0, \quad \text{for all } u \ne 0. \tag{1.7}$$

We will also assume

$$f'(u) > 0 \quad \text{for } 0 < |u| \le \sigma \tag{1.8}$$

for some small $\sigma > 0$. If $f'(0) \ne 0$ then (1.8) is clearly satisfied. Under the assumption (1.8) it turns out that the gradient of a classical solution also cannot blow up in regions where $u_x < 0$.

1.2. Lower Bound for the Gradient. *Assume* (1.7) *and* (1.8). *If u is a classical solution of* (1.4, 2) *on $(-1, 1) \times (0, T)$ then u_x is uniformly bounded on*

$$\Omega = \{(x,t) \in (-1,1) \times (0,T) \mid u_x(x,t) < 0\}.$$

To obtain gradient blow-up we will use traveling wave solutions of (1.4). These are solutions of the form $u(x,t) = \Psi(x - ct)$, where Ψ is any solution of the ODE

$$\Psi'' + c\Psi' + f(\Psi)|\Psi'|^{m-1}\Psi' = 0. \tag{1.9}$$

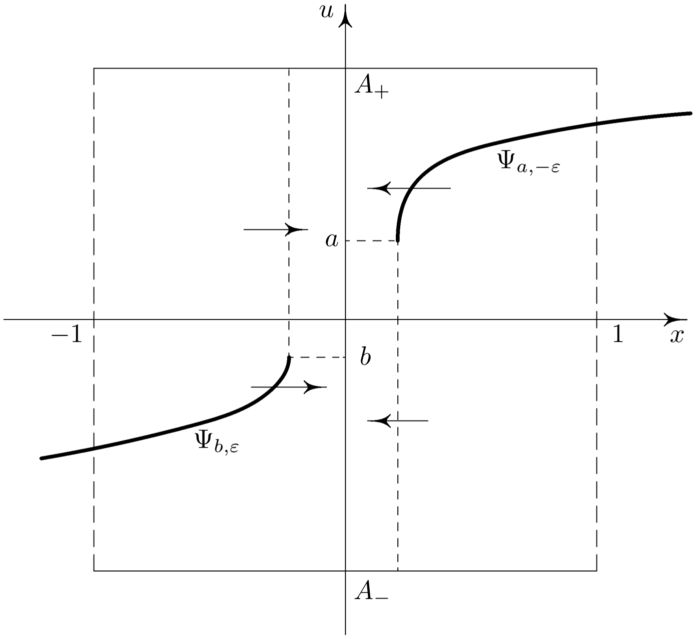
1.3. Singular Traveling Waves. *Assume* (1.7)*. For any $a \ge 0$ and $c \le 0$ there is exactly one solution $\Psi_{a,c}(x)$ defined for $0 \le x < \infty$ of* (1.9) *with*

$$\Psi_{a,c}(0) = a, \qquad \Psi'_{a,c}(0) = \infty. \tag{1.10}$$

This solution is monotonically increasing.

Likewise, for any $a \le 0$ and $c \ge 0$ there exists a unique traveling wave $\Psi_{a,c}(x)$ defined for all $x \in (-\infty, 0]$ which satisfies (1.10)*. This solution is also monotonically increasing.*

We postpone the elementary proof to section 3 and proceed with the construction of our anomalous viscosity sub- and supersolutions.



Colliding Travelling Waves

Let $A_- < 0 < A_+$ be given, and assume that for some $\xi \in (-1, 1)$, $b < 0 < a$ and $\varepsilon > 0$ one has

$$\Psi_{b,\varepsilon}(-\xi - 1) > A_-, \qquad \Psi_{a,-\varepsilon}(-\xi + 1) < A_+. \tag{1.11}$$

If we choose some $T > 0$, and some $B_- \le A_-$ and define

$$v(x,t) = \begin{cases} \Psi_{a,-\varepsilon}(x - \xi + \varepsilon(t - T)) & \text{for } x \ge \xi - \varepsilon(t - T), \\ B_- & \text{for } x < \xi - \varepsilon(t - T). \end{cases} \tag{1.12}$$

then v is a viscosity subsolution of (1.4) on $[-1, 1] \times [0, T]$ in the following sense:

(i) *v is lower semicontinuous,*

(ii) *For any $(x_0, t_0) \in (-1, 1) \times (0, T]$, and any smooth (test)function $\phi(x, t)$ with $\phi(x_0, t_0) = v(x_0, t_0)$ and $\phi \ge v$ on a neighbourhood of (x_0, t_0), one has*

$$\phi_t \le \phi_{xx} + f(v(x_0, t_0))|\phi_x|^{m-1}\phi_x, \qquad \text{at } (x_0, t_0).$$

(iii) *$v(\pm 1, t) \le A_\pm$ for all $0 \le t \le T$.*

In particular, v is also a subsolution in the sense that it obeys a comparison principle with respect to smooth solutions of (1.4): If $u(x,t)$ is any classical solution of (1.4) with boundary conditions (1.2), and initial condition $u(x,0) \geq v(x,0)$ for all $x \in [-1,1]$, then $u(x,t) \geq v(x,t)$ throughout $[-1,1] \times [0,T]$.

Similarly the function

$$w(x,t) = \begin{cases} \Psi_{b,\varepsilon}(x - \xi - \varepsilon(t-T)) & \text{for } x \leq \xi + \varepsilon(t-T), \\ B_+ & \text{for } x > \xi + \varepsilon(t-T) \end{cases} \tag{1.13}$$

with $B_+ \geq A_+$ is a viscosity supersolution, and it obeys a comparison principle with respect to smooth solutions of (1.4).

These sub- and supersolutions violate the comparison principle. At $t = 0$ they are clearly ordered, with $v(x,0) < w(x,0)$ for all $x \in [-1,1]$, but at $t = T$ they are not, since one has $v(\xi,T) = a > 0 > b = w(\xi,T)$. For $t > T$ we get an even more flagrant violation of the maximum principle. The definitions (1.12) and (1.13) still define sub- and supersolutions for $(x,t) \in [-1,1] \times [0,T']$, as long as $T' > T$ satisfies the boundary conditions $v(\pm 1,t) \leq A_\pm$, and $w(\pm 1,t) \geq A_\pm$ for all $t \leq T'$. For any $t \in (T,T']$ one then gets a whole interval $(\xi - \varepsilon(t-T), \xi + (t-T))$ on which v and w are ordered the wrong way around, i.e. $v(x,t) > w(x,t)$.

The bad behaviour of the sub- and supersolutions v and w is directly responsible for gradient blow-up in finite time. While v and w refuse to remain ordered, they do obey the maximum principle when compared to any smooth solution u of (1.4, 2). Thus if u is a smooth solution with $v(x,0) \leq u(x,0) \leq w(x,0)$ for $|x| \leq 1$, then $v \leq u \leq w$ must hold everywhere. This obviously cannot hold for any $t \geq T$, so the classical solution u cannot exist beyond $t = T$.

If one chooses $T = (1+|\xi|)/\varepsilon$, then the initial values of v and w are $v(x,0) \equiv B_-$, and $w(x,0) \equiv B_+$. Given an arbitrary initial condition $u(x,0)$, we can choose $B_- < \min_x u(x,0)$ and $B_+ > \max_x u(x,0)$. Comparison with v and w then shows that the corresponding classical solution must break down before $t = T$, say at $t = T'' < T$. By the maximum principle u remains bounded by $B_- \leq u \leq B_+$, so the gradient u_x must become unbounded as $t \uparrow T''$. To prove that this happens in the interior of the interval $-1 \leq x \leq 1$ and not at the boundary, we must show that u_x remains bounded on the strips $[-1,-1+\delta] \times [0,T'']$ and $[1-\delta,1] \times [0,T'']$. We do this in section 4.

This leads us to the following sufficient condition for interior gradient blow-up.

1.4. Proposition. *Let* (1.7) *hold. Assume* $A_\pm$ *are such that for some* $\xi \in (-1,1)$*, some* $b < 0 < a$ *and* $\varepsilon > 0$ *one has* (1.11)*. Then any classical solution* $u(x,t)$ *of* (1.4, 2) *with* C^1 *initial data* φ *satisfying* $\varphi(\pm 1) = A_\pm$ *has interior gradient blow-up before* $t = T = (1+|\xi|)/\varepsilon$.

After taking a closer look at the traveling waves one can replace the hypothesis on $A_\pm$ in proposition 1.4 by the following more concrete statement: $A_- < 0 < A_+$ must be such that

$$\int_{A_-}^{A_+} \{(m-2)F(y)\}^{\frac{1}{m-2}}\, \mathrm{d}y > 2 \tag{1.14}$$

holds, where $F(y) = \int_0^y f(s)\mathrm{d}s$. In section 3 we show that (1.14) implies that (1.11) is satisfied for some $\xi \in (-1,1)$ and sufficiently small $b < 0 < a$ and $\varepsilon > 0$. We also show in section 3 that (1.14) holds exactly when the prescribed boundary values are such that there is no steady state of (1.4) satisfying them. In view of all this we can restate our sufficient condition for interior blow up as follows:

1.5. Interior Gradient Blow-up Theorem. *Assume* (1.7) . *If* (1.4, 2) *has no steady states, then the gradient of any classical solution* $u(x,t)$ *must blow up in finite time, in the interior of the interval* $(-1,1)$.

This criterion is obviously sharp in the sense that any steady state constitutes a classical solution whose derivative remains uniformly bounded for all time. However, if (1.4, 2) admits steady states one still expects that sufficiently "large" initial data will produce solutions that blow up in finite time.

The plan of the paper is as follows. In section 2 we prove the estimate 1.1. In section 3 we construct the singular traveling waves. In section 4 we show that u_x cannot blow up near the boundary. Proposition 1.2 is proved in section 5. In section 6 we establish the estimate

$$(\Phi(u(x,t)))_x \le C \quad \text{in } [-1,1]\times[0,T), \qquad \Phi(u) = \int_0^u \{(m-2)F(s)\}^{1/(m-2)}\,\mathrm{d}s \tag{1.15}$$

under an additional hypothesis on $f(u)$ near $u=0$. In section 7 it is shown that monotone solutions are at blow-up time at least as singular as the singular traveling waves. Together with (1.15) this implies that the singularities are the same and (1.15) is sharp.

2. Proof of the gradient estimate 1.1

We may assume $f(u) > 0$ in Q_R otherwise we would consider $\tilde{u}(x,t) = u(-x,t)$. We shall apply the maximum principle to the quantity $w = \eta(x,t)u_x^2 e^{-Ku}$, for suitably chosen constant $K \in \mathbf{R}$ and cut-off function $0 \le \eta \in C^2(Q_R)$.

Let $\mathcal{M}$ be the parabolic operator

$$\mathcal{M}[\phi] = \phi_t - \phi_{xx} - m|u_x|^{m-1}f(u)\phi_x.$$

Then

$$\mathcal{M}[u_x^2] = 2u_x\mathcal{M}[u_x] - 2\left(u_{xx}\right)^2 = 2f'(u)|u_x|^{m+1}u_x - 2\left(u_{xx}\right)^2,$$

so that

$$\begin{aligned}\mathcal{M}[\eta u_x^2] &= \eta\mathcal{M}[u_x^2] + u_x^2\mathcal{M}[\eta] - 2\eta_x\left(u_x^2\right)_x \\ &\le \eta\mathcal{M}[u_x^2] + u_x^2\left(\mathcal{M}[\eta] + \frac{4\eta_x^2}{\eta}\right) + \eta u_{xx}^2 \\ &\le 2\eta f'(u)|u_x|^{m+1}u_x + u_x^2\left(\mathcal{M}[\eta] + \frac{4\eta_x^2}{\eta}\right).\end{aligned}$$

Apply the inequality $a^\theta b^{1-\theta} \le \theta a + (1-\theta)b \le a+b$ to the second term, with $\theta = 2/(m+2)$ to get

$$\mathcal{M}[\eta u_x^2] \le (2f'(u)+1)\eta|u_x|^{m+1}u_x + \left\{\eta^{-2/(m+2)}\mathcal{M}[\eta] + \eta^{-1-2/(m+2)}\eta_x^2\right\}^{\frac{m+2}{m}}. \tag{2.1}$$

By choosing $\eta = \zeta^n$, with

$$\zeta(x,t) = \left(1 - \left(\frac{x-x_0}{R}\right)^2\right)(t-t_0)$$

and n a large enough integer, one can ensure the last term in (2.1) is bounded, so that we have

$$\mathcal{M}[\eta u_x^2] \le (2f'(u)+1)\eta|u_x|^{m+1}u_x + C(R). \tag{2.2}$$

For e^{Ku} we have the following equation

$$\begin{aligned}\mathcal{M}[e^{Ku}] &= Ke^{Ku}\mathcal{M}[u] - K^2e^{Ku}u_x^2\\ &= -K(m-1)f(u)|u_x|^{m-1}u_xe^{Ku} - K^2u_x^2e^{Ku}.\end{aligned}$$

The quotient rule for parabolic operators,

$$\mathcal{M}\left[\frac{\phi}{\psi}\right] - 2\frac{\psi_x}{\psi}\left(\frac{\phi}{\psi}\right)_x = \frac{\psi\mathcal{M}[\phi] - \phi\mathcal{M}[\psi]}{\psi^2},$$

when applied to $\phi = \eta u_x^2$ and $\psi = e^{Ku}$ gives for $w = \eta u_x^2 e^{-Ku}$

$$\begin{aligned}&\mathcal{M}[w] - 2Ku_xw_x \leq\\ &\qquad e^{-Ku}\left\{C(R) + \left(2f'(u) + 1 + (m-1)Kf(u)\right)\eta|u_x|^{m+1}u_x + K^2\eta u_x^4\right\}.\end{aligned} \tag{2.3}$$

Since $f(u)$ is bounded away from zero on Q_R we can choose $K \leq 0$ so that

$$(m-1)Kf(u) \leq -2f'(u) - 2$$

on Q_R. Choosing such a K we get in (2.3)

$$\mathcal{M}[w] - 2Ku_xw_x \leq e^{-Ku}\left\{C(R) + \eta(K^2u_x^4 - |u_x|^{m+1}u_x)\right\} \tag{2.4}$$

if $u_x > 0$. On $Q^+ = \{(x,t) \in Q_R \mid u_x(x,t) > 0\}$ we have $K^2u_x^4 - |u_x|^{m+1}u_x \leq C_K$, so together with boundedness of u, (2.4) implies

$$\mathcal{M}[w] - 2Ku_xw_x \leq C = e^{-K\beta}(C(R) + C_K),$$

and hence, by the maximum principle, $w = \eta u_x^2 e^{-Ku} \leq C(t - t_0 + R^2)$. This implies that u_x is bounded from above on $Q_{R/2}$.

To obtain a lower bound for u_x one chooses $K \geq 0$ so that $(m-1)Kf(u) \geq -2f'(u)$ on Q_R and proceeds as before, starting at (2.3).

■

3. Traveling wave solutions

To solve (1.9) we first assume the solution $y = \Psi(x)$ is monotonically increasing, and consider the inverse function $x = \Phi(y)$. This function is a solution of the ODE

$$\Phi''(y) = c\Phi'(y)^2 + f(y)\Phi'(y)^{3-m}. \tag{3.1}$$

Multiplication with $\Phi'(y)^{m-3}$ gives an ODE for $\Upsilon(y) = \Phi'(y)^{m-2}$,

$$\Upsilon'(y) = (m-2)\left\{c\Upsilon(y)^{1+\frac{1}{m-2}} + f(y)\right\}. \tag{3.2}$$

Since $m > 2$ this ODE has a solution $\Upsilon_{a,c}$ with $\Upsilon_{a,c}(a) = 0$ on a small neighbourhood of $y = a$. Suppose $a \geq 0$ and $c \leq 0$. For $y > a$ one will have $\Upsilon_{a,c}(y) > 0$ because $f(y) > 0$ for all $y > a$. Since $c \leq 0$, the solution $\Upsilon_{a,c}$ can be defined for all $y > a$. Proposition 1.3 follows directly by integrating as follows

$$\Phi_{a,c}(y) = \int_a^y \Upsilon_{a,c}(\eta)^{\frac{1}{m-2}}\mathrm{d}\eta,$$

and inverting $\Phi_{a,c}$ we obtain the traveling wave $\Psi_{a,c}$. Similar arguments apply when $a \le 0$, $c \ge 0$.

If the wave speed c vanishes, so that one is dealing with steady states, (3.2) can be integrated, after which one can integrate $\Phi'(y)$ to obtain

$$\Phi(y) = \int_0^y \{(m-2)F(\eta)\}^{\frac{1}{m-2}}\, \mathrm{d}\eta, \tag{3.3}$$

with $F(y) = \int_0^y f(\eta)\mathrm{d}\eta$.

If we assume that (1.14) holds for some $A_\pm$, then $\Phi(A_+) - \Phi(A_-) > 2$. We can then choose $\xi \in (-1,1)$ so that $\Phi(A_+)+\xi > 1$, and $\Phi(A_-)+\xi < -1$ (e.g. $\xi = (\Phi(A_+) + \Phi(A_-))/2$). Since the traveling waves depend continuously on a and the velocity c, we will still have

$$\Phi_{a,-\varepsilon}(A_+) + \xi > 1, \quad \text{and} \quad \Phi_{b,\varepsilon}(A_-) + \xi < -1 \tag{3.4}$$

for sufficiently small $b < 0 < a$ and $\varepsilon > 0$. The conditions (3.4) are precisely those of (1.11).

4. Interior blow-up

The arguments in the introduction show that a classical solution cannot exist forever, which implies that u_x must blow up in finite time. In this section we show that this cannot happen near the boundary.

4.1. Proposition. *Assume* (1.7). *Let $u(x,t)$ be a classical solution on $Q_T = (-1,1)\times(0,T)$ of* (1.4,2). *Then there are $C, \delta > 0$ such that $|u_x(x,t)| \le C$ for $1-\delta < |x| \le 1$, $0 < t < T$.*

Proof. Let $\psi(x)$ be the solution of

$$\psi''(x) + f(\psi(x))|\psi'(x)|^{m-1}\psi'(x) = 0, \qquad \psi(1) = A_+, \psi'(1) = L > 0.$$

The solution can again be found by integration, and one finds

$$\psi'(x)^{-(m-2)} = L^{-(m-2)} - (m-2)\int_{\psi(x)}^{A_+} f(\eta)\mathrm{d}\eta.$$

¿From this one can deduce that if L is large, ψ is a concave increasing function defined on an interval $[x_1, 1]$, with $\psi'(x_1) = \infty$. The value of $\psi(x_1)$ is determined by

$$L^{-(m-2)} = (m-2)\int_{\psi(x_1)}^{A_+} f(\eta)\mathrm{d}\eta,$$

so that $A_+ - \psi(x_1) = O(L^{-(m-2)}) = o(1)$. Concavity implies $1 - x_1 \le (A_+ - \psi(x_1))/L = o(1)$.

If the constant $L > 0$ is chosen large enough, then $\psi(x)$ will lie below the initial value $\varphi(x)$ for $x_1 \le x \le 1$. Since ψ is a steady state, and since $\psi'(x_1) = +\infty$, the maximum principle implies that $u(x,t) \ge \psi(x)$ for all $x_1 \le x \le 1$, $0 \le t < T$. Hence u is bounded from below for $x_1 \le x \le 1$, and

$$u_x(1,t) \le \psi'(1) = L, \quad \text{for } 0 \le t < T. \tag{4.1}$$

To obtain a lower bound for $u_x(1,t)$ we observe that by the maximum principle

$$u(x,t) \le \mu(1-x) + A_+, \qquad \mu = \max\{0, -\inf \varphi'\}, \tag{4.2}$$

hence $u_x(1,t) \geq -\mu$.

To show that u_x must remain bounded on $[x_2, 1]\times[0,T)$, $x_2 = \frac{1+x_1}{2}$, we slightly modify the argument from section 2. We apply the maximum principle to $w = \eta u_x^2 e^{-Ku}$, $\eta(x) = \left(\frac{x-x_1}{1-x_1}\right)^n$, where n and K are chosen as in section 2. Instead of $Q_R(x_0,t_0)$ we now consider $(x_1,1)\times(0,T)$ and we obtain a bound that depends also on $\sup_{[x_1,1]} |\varphi'|$ and $\sup_{[0,T)} |u_x(1,t)|$.

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5. Proof of proposition 1.2

The quantity $v = u_x$ satisfies

$$v_t - v_{xx} - mf(u)|u_x|^{m-1}v_x = f'(u)|v|^{m+1}. \tag{5.1}$$

If we assume that $f'(u) \geq c > 0$ for all u and

$$\Omega = \{(x,t) \mid 0 < t < T,\ a(t) \leq x \leq b(t),\ u_x(a(t),t) \equiv u_x(b(t),t) \equiv 0\},$$

then the maximum principle immediately gives the estimate

$$u_x \geq -(mct)^{-1/m}, \tag{5.2}$$

which is independent of the initial condition.

Proposition 1.2 states that a weaker version of this estimate (5.2) holds for general Ω if one only assumes that $f'(u) > 0$ for sufficiently small $u \neq 0$. For the proof we consider the quantity $v = g'(u)u_x = g(u)_x$, where g is an as yet unspecified function with $g'(u) \geq 0$. A computation shows that v satisfies

$$\mathcal{L}[v] = A(u)|v|^{m+1} + B(u)v^3, \tag{5.3}$$

with

$$\mathcal{L}[\phi] = \phi_t - \phi_{xx} - \left\{mf(u)|u_x|^{m-1} - 2\frac{g''(u)}{g'(u)}u_x\right\}\phi_x, \tag{5.4}$$

$$\begin{aligned} A(u) &= \frac{g'(u)f'(u) - (m-1)g''(u)f(u)}{g'(u)^{m+1}} \\ &= \frac{1}{g'(u)}\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{f(u)}{g'(u)^{m-1}}\right), \end{aligned} \tag{5.5}$$

$$B(u) = \frac{1}{g'(u)}\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left(\frac{1}{g'(u)}\right). \tag{5.6}$$

We shall now choose $g(u)$ so that $A(u) > 0$ for $u \neq 0$, and so that $B(u)/A(u)$ is bounded. To do this we put

$$g'(u) = \begin{cases} 1 & \text{for } |u| \leq \sigma/2, \\ \exp\left(-K(|u| - \sigma/2)^2\right) & \text{for } |u| \geq \sigma/2. \end{cases}$$

Then for $|u| \leq \sigma/2$ we have

$$A(u) = f'(u) \geq 0, \qquad B(u) \equiv 0.$$

For $\sigma/2 \le |u| \le \sigma$ we get $g''/g' = -2K(|u| - \sigma/2)\mathrm{sign}(u)$ so that $f(u)g''(u)/g'(u) \le 0$. Hence

$$\begin{aligned} A(u) &= g'(u)^{-m}\left\{f'(u) - (m-1)\frac{g''(u)}{g'(u)}f(u)\right\} \\ &\ge g'(u)^{-m} f'(u) \\ &> 0. \end{aligned}$$

Let $M = \sup_{[-1,1]\times[0,T)} |u(x,t)|$. For $\sigma \le |u| \le M$ we still have $g''/g' = -2K(|u| - \sigma/2)\mathrm{sign}(u)$, and therefore

$$\begin{aligned} A(u) &\ge g'(u)^{-m}\left\{f'(u) + 2K\left(|u| - \sigma/2\right)|f(u)|\right\} \\ &\ge g'(u)^{-m}\left\{f'(u) + 2K\sigma|f(u)|\right\}. \end{aligned}$$

If we choose $K = \sup_{\sigma\le|u|\le M} |f'(u)/\sigma f(u)|$ then $A(u)$ will also be bounded from below for $\sigma \le |u| \le M$.

The lower estimates for $A(u)$ and the boundedness of $B(u)$ together imply that $|B(u)| \le C_0 A(u)$ for some constant $C_0 < \infty$.

We now apply the maximum principle to v on the domain

$$\Omega = \{(x,t) \in (-1,1)\times(0,T) \mid u_x(x,t) < 0\}.$$

On the part of the parabolic boundary of Ω where $0 < t < T$ one either has $u_x = 0$ and hence $v = 0$, or one has $|x| = 1$. From (4.2) it follows that $v(1,t) \ge \inf\varphi'$ and if $t = 0$ then v is also bounded by $\inf\varphi'$. In Ω v satisfies

$$\mathcal{L}[v] \ge A(u)\left(|v|^{m+1} - C_0|v|^3\right)$$

so that the maximum principle implies

$$v \ge \min\left(-C_0^{1/(m-2)}, \inf\varphi'\right).$$

Since $u_x = v/g'(u) \ge v/g'(M)$ we get the desired lower bound for u_x.

■

6. Estimate (1.15)

6.1. Proposition. *Assume that f satisfies* (1.7), *and that for some small $\sigma > 0$ one has*

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{f(u)}{F(u)^{m-1}}\right) < 0 \quad \text{for } 0 < |u| \le \sigma. \tag{6.1}$$

Then there is a constant $C > 0$ such that (1.15) *holds.*

The assumption (6.1) is satisfied for all $u \ne 0$ if $f(u) = |u|^{q-1}u$, $q \ge 1$.

Proof. To obtain an upper bound for $\Phi(u)_x$ we again consider $v = g(u)_x = g'(u)u_x$ and choose g appropriately. Thus we will obtain an estimate for $\Phi(u)_x$ on the region

$$\Sigma = \{(x,t) \mid u(x,t) > 0\}.$$

By applying the same arguments to $\hat{u}(x,t) = -u(-x,t)$, which satisfies

$$\hat{u}_t = \hat{u}_{xx} - f(-\hat{u})|\hat{u}_x|^{m-1}\hat{u}_x$$

one obtains an estimate for $\Phi(u)_x$ in the region where $u < 0$.

We first present the proof under the assumption that (6.1) holds for all $u \neq 0$, i.e. for $\sigma = \infty$, and then indicate how to modify the proof to deal with the more general case $\sigma < \infty$.

In the previous section we saw that v satisfies (5.3), with $\mathcal{L}$, $A(u)$ and $B(u)$ as in (5.4–6). Choose g so that $g'(0) = 0$, $B(u) = -A(u)$, and require that $A(u) < 0$ for all u. If we do this, then v will satisfy

$$\mathcal{L}[v] = A(u)\left(|v|^{m+1} - v^3\right),$$

in which the left hand side is negative whenever $v > 1$. On the part of the parabolic boundary of Σ where $0 < t < T$ we either have $v = 0$, or we have $x = 1$ and then $v(1,t) \le g'(A_+)L$, L is from (4.1). Since $v(x,0) \le \sup_{|x|\le 1} |(g(\varphi(x)))'|$, the maximum principle implies that v is uniformly bounded by

$$v \le \max\left\{1,\ g'(A_+)L,\ \sup_{|x|\le 1} |(g(\varphi(x)))'|\right\},$$

as claimed.

To complete the proof in the case $\sigma = \infty$ we must show that we can choose g so that $B = -A > 0$ holds. The relation $B + A = 0$ implies the following ODE for $g'(u)$:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{g'(u)}\right) + \frac{f(u)}{g'(u)^{m-1}} + c = 0,$$

for some constant $c \in \mathbf{R}$. This equation can be rewritten as

$$g''(u) = cg'(u)^2 + f(u)g'(u)^{3-m},$$

which is exactly the "inverse of a traveling wave" equation (3.1). We may therefore choose $g(u) = \Phi_{0,c}(u)$, with $c \ge 0$, provided the resulting B will be positive. If one chooses $c = 0$, then $\Phi(u) = g(u)$ is explicitly given in terms of f by (3.3). Substitution of (3.3) in our expression for A then shows that $A(u) < 0$ is equivalent with (6.1). This completes the proof if one assumes (6.1) holds with $\sigma = \infty$.

We now turn to the situation where σ is finite. We choose $g(u)$ so that $g(u) = \Phi(u)$ for small values of u. For arbitrary $u > 0$ we let $g(u)$ be determined by the requirement that

$$\frac{g''(u)}{g'(u)} = \frac{\Phi''(u)}{\Phi'(u)} + \eta_\varepsilon(u - \sigma/2)\left(K - \frac{\Phi''(u)}{\Phi'(u)}\right)$$

holds for $u > 0$. Here η_ε is a smooth nondecreasing function with $\eta_\varepsilon(s) \equiv 0$ for $s \le \varepsilon$ and $\eta_\varepsilon(s) \equiv 1$ for $s \ge 2\varepsilon$. We choose $\varepsilon < \sigma/2$, and K is any constant satisfying

$$K > \sup\left\{\frac{\Phi''(u)}{\Phi'(u)} \mid |u - \sigma/2| \le \varepsilon\right\} \qquad \text{and}$$

$$K > \sup\left\{\frac{f'(u)}{(m-1)f(u)} \mid \sigma \le u \le M\right\},$$

where $M = \sup\{u(x,t) \mid 0 < t < T, |x| < 1\}$.

For $0 < u \le \sigma$ we then have $g''/g' \ge \Phi''/\Phi'$, so that

$$A(u) < g'(u)\left\{f'(u) - (m-1)\frac{\Phi''(u)}{\Phi'(u)}f(u)\right\} < 0.$$

For $\sigma \le u \le M$ we get

$$A(u) < g'(u)\left\{f'(u) - (m-1)Kf(u)\right\} < 0.$$

For $0 < u < \sigma/2 - \varepsilon$ we also have $A(u) + B(u) = 0$, and since $B(u)$ is bounded, there is some $C_1 < \infty$ such that $|B(u)| \le -C_1 A(u)$ for all $0 < u \le M$. We can now apply the maximum principle to v on Σ. From

$$\mathcal{L}[v] \le A(u)\left(|v|^{m+1} - C_1|v|^3\right) \qquad \text{on } \Sigma$$

we see that v is bounded by the larger of $C_1^{1/(m-2)}$ and $\sup v(x,0) = \sup_x \Phi(u)_x(x,0)$. This implies a similar bound for $\Phi(u)_x$ since $\Phi(u) = \Phi \circ g^{-1}(v)$, and $\Phi \circ g^{-1}$ is C^1 by construction.

■

7. Limiting profile

We assume in this section that the initial data φ is monotone, i.e. $\varphi'(x) > 0$ for $|x| \le 1$, and that u_x blows up at the time $t = T$.

7.1. Proposition. *The solution $u(x,t)$ must be monotone for all $0 \le t < T$, and u_x is bounded away from zero on $Q_T = (-1,1) \times (0,T)$.*

Proof. Differentiate (1.4), and you find that $p = u_x$ satisfies

$$p_t = p_{xx} + m|u_x|^{m-1}f(u)p_x + f'(u)|u_x|^{m-1}u_x p \tag{7.1}$$

on $Q_T = (-1,1) \times (0,T)$. From $A_- \le u \le A_+$ we get that $p \ge 0$ on the parabolic boundary of Q_T, so the maximum principle implies that $p > 0$ on Q_T.

To bound $p = u_x(1,t)$ away from zero one considers the solution $\psi(x,\delta)$ of

$$\psi'' + f(\psi)|\psi'|^{m-1}\psi' = 0, \qquad \psi(1) = A_+, \psi'(1) = \delta.$$

For $\delta = 0$ the solution is $\psi(x) \equiv A_+$. For small $\delta > 0$ the solution ψ will still be defined on the interval $[-1,1]$, and it will be close to the constant solution $\psi \equiv A_+$. Hence it will dominate the initial condition, and since $\psi(x,\delta)$ is a steady state, it must dominate $u(x,t)$ at all times $t < T$. This implies $u_x(1,t) \ge \psi'(1) = \delta$. A similar argument provides a uniform lower bound for $u_x(-1,t)$. The maximum principle applied to p again yields a lower bound for p on Q_T.

■

Denote the level curves of $u(x,t)$ by $X(\eta,t)$, thus

$$u(X(\eta,t),t) \equiv \eta. \tag{7.2}$$

By the implicit function theorem $X(\eta,t)$ is a C^1 function for $A_- \le \eta \le A_+$, $0 \le t < T$.

7.2. Proposition. *If the initial data φ are C^2, then X_η and X_t are uniformly bounded, and hence $\lim_{t\uparrow T} X(\eta,t)$ exists uniformly in $\eta \in [A_-, A_+]$.*

Proof. Since $X_\eta = u_x^{-1}$ the lower bound on u_x implies an upper bound on X_η.

The quantity $q(x,t) = u_t$ satisfies the same equation (7.1) as p, and vanishes at the endpoints $x = \pm 1$. Let

$$c = \sup_{x\in[-1,1]} \frac{|q(x,0)|}{p(x,0)} = \sup_{x\in[-1,1]} \left| \frac{\varphi''(x)}{\varphi'(x)} + f(\varphi(x))(\varphi'(x))^{m-1} \right|. \tag{7.3}$$

Then the maximum principle implies that $|q(x,t)| \le cp(x,t)$ on Q_T.

In view of $X_t = -u_t/u_x$ we have shown that $|X_t| \le c$.

■

A second proof of boundedness of X_t can be given by observing that $X(\eta,t)$ satisfies the quasilinear equation

$$X_t = \frac{X_{uu}}{X_u^2} - f(u)X_u^{1-m}. \tag{7.4}$$

Differentiation in the time direction then yields

$$X_{tt} = \frac{X_{tuu}}{X_u^2} + \left\{ (m-1)f(u)X_u^{-m} - \frac{2X_{uu}}{X_u^3} \right\} X_{tu}.$$

As $X_t(A_\pm, t) \equiv 0$ the maximum principle shows that X_t is bounded by its initial values.

7.3. Corollary. *Assume* (1.7). *Let $\xi = \lim_{t\uparrow T} X(0,t)$. Then $u_x(x,t)$ is uniformly bounded as $t \uparrow T$ outside any small interval $(\xi - \delta, \xi + \delta)$; and $u(x,t)$ converges uniformly outside any such interval. Moreover, the limit $u(x,T)$ is continuous provided* (6.1) *holds.*

Proof. The previous proposition implies $f(u(x,t))$ is uniformly bounded away from zero on $[\xi + \delta, 1] \times [T - \tau, T)$ for any $\delta > 0$ and sufficiently small $\tau > 0$. Since $u_x(1,t)$ is uniformly bounded by proposition 4.1, the interior gradient bounds of section 2 imply u_x is indeed bounded on $[\xi + \delta, 1] \times [T - \tau, T)$.

Proposition (6.1) together with boundedness of X_t yield uniform continuity of $u(x,t)$ on $[-1,1] \times [0,T)$.

■

The following proposition shows that estimate (1.15) is optimal.

7.4. Proposition. *Assume* (1.7). *If $\varphi \in C^2$ and c is as in* (7.3), *then*

$$u(x,T) \ge \Psi_{0,c}(x - \xi) \quad for\ x > \xi.$$

Proof. ¿From $|X_t| \le c$ and (7.4) we obtain

$$X_u^{m-3} X_{uu} \le f(u) + cX_u^{m-1},$$

which implies that $Y = X_u^{m-2}$ satisfies

$$Y_u \le (m-2) \left\{ f(u) + cY^{1+1/(m-2)} \right\}. \tag{7.5}$$

Observe that (7.5) is the same as (3.2) with inequality instead of equality. Let $x_n \to \xi$ and $t_n \uparrow T$ be such that $u_x(x_n, t_n) \to \infty$. Let Y_n be the solution of (3.2) with $Y_n(\eta_n) = Y(\eta_n, t_n)$, $\eta_n = u(x_n, t_n)$. Then $Y(\eta, t_n) \le Y_n(\eta)$ for $0 < \eta < A_+$. After integrating this we get

$$X(\eta, t_n) \le X(0, t_n) + \int_0^\eta Y_n(\rho)^{1/(m-2)} \mathrm{d}\rho. \tag{7.6}$$

As $n \to \infty$ one has $Y_n(\eta) \to \Upsilon_{0,c}(\eta)$, so (7.6) implies $X(\eta, T) \le \xi + \Phi_{0,c}(\eta)$ and thus $u(x, T) \ge \Psi_{0,c}(x - \xi)$, as claimed.

■

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