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THE RADIUS OF VANISHING BUBBLES IN EQUIVARIANT HARMONIC MAP FLOW FROM D^2 TO S^2*

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Abstract. We derive an upper bound for the radius $R(t)$ of a vanishing bubble in a family of equivariant maps $F_t : D^2 \to S^2$ which evolve by the harmonic map flow. The self-similar "type 1" radius would be $R(t) = C\sqrt{T-t}$. We prove that $R(t) = o(T-t)$.

Key words. harmonic map flow, asymptotics, singularities

AMS subject classifications. 35K55, 53C44

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1. Introduction.

Let $N^n \subset \mathbb{R}^k$ be a smooth submanifold. The Dirichlet integral or *energy* of a map F from the unit disc $D^2 \subset \mathbb{R}^2$ into N is defined to be

$$\mathfrak{D}[F] = \frac{1}{2}\int_{D^2} |\nabla F(x)|^2.$$

Extremals of this energy with prescribed boundary values $F|_{\partial D^2}$ are called *harmonic maps*. Eells and Sampson [4] introduced the gradient flow for $\mathfrak{D}[F]$, now called the *harmonic map flow*, in which a family of maps $F_t : D \to N$ evolves according to the nonlinear heat equation

$$\frac{\partial F}{\partial t} = \big(\Delta F\big)^{\mathsf{T}}. \tag{1.1}$$

Here, for any point $p \in N$ and vector $v \in T_p\mathbb{R}^k$, we write v^{T} for the tangential component of v to T_pN.

When the target N is the two-dimensional sphere, the harmonic map flow has recently appeared as a model for the direction field of a nematic liquid crystal; see [12] where the motivation comes from applications in fiber spinning, but physical applications go back as far as the treatment of ferromagnetic materials by Landau and Lifschitz [5].

For general targets (1.1) has been used in a purely mathematical context to construct harmonic maps of a given homotopy type; see, e.g., [7]. As a nonlinear vector-valued partial differential equation, the harmonic map flow is of interest because of the possible formation of singularities, due to the presence of topological obstructions.

For targets with negative sectional curvatures Eells and Sampson showed that the initial value problem for (1.1) has a unique global solution $\{F_t \mid t \geq 0\}$, which

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converges to a harmonic map as $t \nearrow \infty$. Struwe [8, 6] later constructed global solutions for arbitrary targets N, which he allowed to have singularities at a finite number of points in space-time $D \times [0, \infty)$. That such singularities cannot be avoided was shown by examples of Chang, Ding, and Ye [2] as well as Coron and Ghidaglia [3]. For a nice treatment of their appearance and possible disappearance in the case of $N = S^2$ with radial symmetry, see [1].

Based on work of Struwe, Ding, Qing, Tian, Topping, and others (see Topping's papers [10, 9] and the references therein), one can give a good qualitative description of Struwe's solutions near their singular points. This description implies that whenever a singularity occurs a harmonic map $f : S^2 \to N$ "bubbles off," i.e., for a singular point $(a, T) \in D \times (0, \infty)$ there exist times $t_i \nearrow T$, points $a_i \to a$, and scales $R_i \searrow 0$, as well as a nonconstant harmonic map $f : S^2 \to N$ such that

$$F_{t_i}(a_i + R_i x) \to f \circ \sigma(x) \qquad (i \to \infty)$$

uniformly in x on compact subsets of $\mathbb{R}^2$. Here $\sigma : \mathbb{R}^2 \to S^2 \setminus \{p\}$ is the inverse of stereographic projection from the point $p \in S^2$. A full description involves the combination and/or superposition of several such "bubbles" (see [9, 10]).

One can now ask at what rate the bubbles vanish, i.e., *how large are the scales R_i relative to the time to blow-up $T - t_i$?* The natural scale, suggested by the parabolic equation, would be $R_i^2 \approx C(T - t_i)$, but this can be ruled out. In fact, Topping [10] has shown that one always has

$$R_i^2 = o\left(\frac{T - t_i}{|\ln(T - t_i)|}\right), \tag{1.2}$$

along some sequence $t_i \nearrow T$, while he also constructed a compact C^∞ smooth target manifold N and a solution $F : D \times [0, T) \to N$ with

$$\liminf_{i\to\infty} R_i^2 (T - t_i)^{-1-\delta} > 0 \tag{1.3}$$

for any $\delta > 0$, thus showing that the upper bound (1.2) cannot be improved in general.

In this note we consider the special case where the target N is the perfectly round two-sphere $S^2 \subset \mathbb{R}^3$ and where the maps $F^t : D^2 \to S^2$ have rotational symmetry, i.e., the case studied in [2, 3, 1]. A later detailed analysis using formal matched asymptotic expansions by van den Berg, Hulshof, and King [11] strongly suggests that a variety of blow-up rates are possible, depending on the specified initial and boundary data. None of the formal solutions in [11] satisfy Topping's lower bound (1.3). In fact, the "generic case" in [11] has

$$R(t) \sim \kappa \, \frac{T - t}{\big(\ln(T - t)\big)^2} = o(T - t) \tag{1.4}$$

for some constant $\kappa > 0$ which varies from solution to solution.

Our main result here is a rigorous example of a solution to harmonic map flow for which we can give an upper bound for the blow-up rate of the radii R_i.

THEOREM 1.1. *There exist a solution $F : D \times [0, T) \to S^2$ which forms a singularity at the origin at time T and a decreasing function $R : [0, T) \to \mathbb{R}_+$ such that*

$$\lim_{t\nearrow T} F^t(R(t)x) = \sigma(x) \tag{1.5}$$

uniformly on compact subsets of $\mathbb{R}^2$. *The length scale* R *satisfies*

$$R(t) = o(T - t) \qquad (t \nearrow T) \tag{1.6}$$

and also an integrated version of this estimate

$$\int_0^T \frac{R(t)\, dt}{(T-t)^2} < \infty. \tag{1.7}$$

In fact, we will derive the estimate for any solution F whose initial data satisfies a certain monotonicity condition (2.5a), (2.5b). In section 5 we note that such initial data are easily constructed.

Note that our upper estimates for the length scale $R(t)$ is less than Topping's generally valid estimate (1.2) by a factor $(T - t_i)^{1/2+o(1)}$, while it differs only from the formal asymptotics (1.4) by logarithmic factors. This raises the question of which of the two behaviors (1.2) and (1.4) is more common: is the blow-up rate (1.4) exceptional and possible only in situations with a high degree of symmetry, or do most singular solutions of harmonic map flow blow up according to (1.4)?

Outline of the paper. We begin by describing the class of symmetric initial data we consider, and recall from the general theory that they do indeed produce solutions with finite time singularities. We establish a number of monotonicity properties of the solutions. Then, using the Sturmian theorem on intersections of solutions to parabolic equations in one dimension, we show that a bubble forms as $t \nearrow T$. This proof also gives us a quantitative estimate (Lemma 6.2 in section 6) on how close the singular bubble at time t is to an actual harmonic map, and this leads to a weaker form of the lower bound for $R(t)$ in the theorem. In the end a careful analysis of the parabolic blow-up of the solution allows us to improve this estimate to $R(t) = o(T - t)$.

2. A class of solutions with symmetry.

We describe here the class of solutions to which our estimates apply.

Rotational symmetry. When the target manifold is $N = S^2 \subset \mathbb{R}^3$, the normal component of ΔF is $-|\nabla F|^2 F$ so that (1.1) becomes

$$\frac{\partial F_t}{\partial t} = \Delta F_t + |\nabla F_t|^2 F_t. \tag{2.1}$$

We choose spherical coordinates (θ, φ) on S^2 and consider maps of the form

$$F_t(r, \theta) = (\cos\theta \sin\varphi(r,t),\, \sin\theta\sin\varphi(r,t),\, \cos\varphi(r,t)). \tag{2.2}$$

Direct computation shows that harmonic map flow (2.1) preserves this class of maps and is equivalent to the following PDE for φ:

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} - \frac{\sin 2\varphi}{2r^2}, \tag{2.3}$$

where $0 \le r \le 1$.

For φ close to zero the last nonlinear term in (2.3) may be approximated by the linear term $-\varphi/r^2$. The resulting linear equation has a singularity in $r = 0$, which forces bounded solutions to have a first order zero in $r = 0$. This property of bounded solutions will result in the boundary condition at $r = 0$ for φ below.

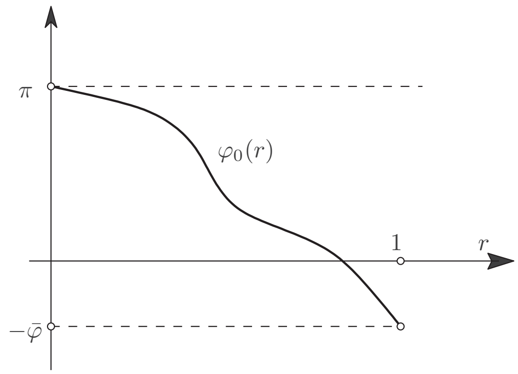

FIG. 2.1. *The initial data.*

The singularity at $r = 0$ in (2.3) is caused by the use of spherical coordinates in the target and polar coordinates in the domain. When we approximate $\sin\phi$ by ϕ, we replace the target S^2 by its tangent plane, described in polar coordinates with ϕ (or $\phi - k\pi$ for some integer k) acting as radius and θ as angular coordinate. All bounded solutions of (2.3) will have the property that for some k the function $\phi(r,t) - k\pi$ has a first order zero in $r = 0$. The solution wants to preserve this value of k. The singularities we are concerned with in this paper are forced to occur when this is no longer possible.

Next we choose initial and boundary conditions for φ ensuring the occurence of a singularity. Choose some $0 < \overline{\varphi} < \pi$ and consider harmonic map flows given by (2.2), where $\varphi : [0,1] \times [0,T)$ satisfies

$$\varphi(1,t) = -\overline{\varphi}, \qquad \varphi(0,t) = \pi. \tag{2.4}$$

We shall assume that at time $t = 0$ one has

$$\varphi(r,0) \le \pi, \tag{2.5a}$$

$$\varphi_{rr} + \frac{1}{r}\varphi_r - \frac{\sin 2\varphi}{2r^2} < 0 \tag{2.5b}$$

for all $r \in [0,1]$.

Henceforth, $\varphi : [0,1] \times [0,T) \to \mathbb{R}$ denotes the corresponding maximal classical solution to (2.3) with boundary conditions (2.4).

In section 5 we show that initial data $\varphi(r,0)$ satisfying the hypotheses (2.5a) and (2.5b) actually do exist (see Figure 2.1). It follows from the work of Chang, Ding, and Ye [2] that any solution whose initial data satisfy (2.5a) and (2.5b) will indeed become singular in finite time.

Monotonicity properties. In section 3 we will use the maximum principle to prove the following.

LEMMA 2.1. *$\varphi_t(r,t) < 0$ for all $(r,t) \in (0,1) \times [0,T)$.*

One could try to use the maximum principle to show that $\varphi_r < 0$ is also preserved by the flow. However, this turns out to be a consequence of the condition $\varphi_t < 0$, which we have imposed on our initial data.

LEMMA 2.2. *$\varphi_r(r,t) < 0$ for all $(r,t) \in (0,1) \times [0,T)$.*

See section 5 for the proof.

The radius of the bubble. Because of (2.4) and $\varphi_r < 0$, there is a unique $R(t) \in (0,1)$ for each $t \in [0,T)$ such that

$$\varphi(R(t),t) = \pi/2, \tag{2.6}$$

i.e., the corresponding map F^t maps the circle in D^2 with radius $R(t)$ to the equator in S^2. By the implicit function theorem, $R(t)$ is a monotonically decreasing function of time with

$$R'(t) = -\varphi_t(R(t),t)/\varphi_r(R(t),t).$$

The radius $R(t)$ defined here is the one we meant in Theorem 1.1.

For any initial function φ_0 satisfying our hypotheses (2.5a), (2.5b) there exist small $\varepsilon > 0$ and large $T_* > 0$ such that the Chang, Ding, and Ye supersolution $\Phi^{\varepsilon,T_*}(\cdot,0)$ lies above φ_0 at $t=0$. By the maximum principle, this continues to hold for $t>0$, and, as argued in [2], the solution φ must become singular before $t = T_*$.

Suppose that the solution becomes singular at time $T < T_*$; then along some sequence of times $t_i \nearrow T$ and points $p_i \in D^2$, a blow-up of the maps F^t will result in a nontrivial harmonic map from $\mathbb{R}^2 \to S^2$. The limit map will inherit the symmetries of the maps F^{t_i}. Because of this, the only possible blow-up point is the origin, and the only possible blow-up map is inverse stereographic projection. We therefore conclude from the general theory that along some sequence $t_i \nearrow T$ one has $R(t_i) \searrow 0$, and

$$\lim_{i\to\infty} \varphi(R(t_i)z, t_i) = \pi - 2\arctan z.$$

Since the bubble radius $R(t)$ is a monotone function of time, we immediately have the following stronger statement.

LEMMA 2.3. *The maximal classical solution φ becomes singular in finite time, i.e., $T<\infty$. Moreover,* $\lim_{t\nearrow T} R(t) = 0$.

In Lemma 6.2 we will show that the bubble forms for all t close to T instead of just along a sequence $t_i \nearrow T$.

LEMMA 2.4. *One has*

$$\lim_{t\nearrow T} \varphi(R(t)z, t) = \pi - 2\arctan z$$

uniformly on bounded z intervals.

3. Proof of Lemma 2.1. It will be convenient to abbreviate

$$f(\varphi) = \frac{1}{2}\sin 2\varphi = \sin\varphi\cos\varphi.$$

We consider $u = \varphi_t$ and $v = u(r,t)/r$. For u one computes

$$u_t = u_{rr} + \frac{1}{r}u_r - \frac{f'(\varphi(r,t))}{r^2}u.$$

From this one obtains

$$v_t = v_{rr} + \frac{3}{r}v_r + \frac{1-f'(\varphi(r,t))}{r^2}v.$$

Since φ comes from a classical solution of harmonic map flow, we have

$$|\varphi_t(r,t)| = \left|\partial_t F^t(r,\theta)\right|,$$

where the right-hand side actually does not depend on θ. For arbitrary $\delta > 0$ the map F^t is smooth on $D^2 \times [0, T-\delta]$, so we have $|\partial_t F^t| \le Cr$ (the constant C may depend on δ). Consequently, $v(r,t) = \varphi_t(r,t)/r$ is uniformly bounded for $0 < r < 1$, $0 \le t \le T - \delta$.

We also may conclude from the smoothness of F_t, i.e. from the boundedness of $|\nabla F_t|$, that

$$|\varphi(r,t) - \pi| \le C_\delta r \quad \text{for} \quad 0 < r < 1, \quad 0 \le t \le T - \delta,$$

and, hence,

$$|1 - f'(\varphi(r,t))| = |1 - \cos\varphi(r,t)| \le \varphi^2 \le C_\delta^2 r^2.$$

Thus v satisfies

$$v_t = v_{rr} + \frac{3}{r} v_r + Q(r,t)v, \tag{3.1}$$

where $Q(r,t) = r^{-2}(1 - f'(\varphi(r,t)))$ is uniformly bounded.

The differential operator in (3.1) is the radial Laplacian in $\mathbb{R}^4$ with a bounded potential added, so the weak maximum principle holds. Therefore, $v(r,0) < 0$ (assumption (2.5b)) implies $v(r,t) \le 0$ for $0 \le r \le 1$ and $0 < t < T$. The strong maximum principle then implies $v(r,t) < 0$ and hence $\varphi_t < 0$ for $0 < r < 1$ and $0 < t < T$.

4. The x variable and the energy E. Instead of considering (2.3) we change the independent variable r to $x = -\ln r$ (so $0 < r < 1$ implies $x > 0$) and study the PDE

$$\varphi_t = e^{2x}\big(\varphi_{xx} - f(\varphi)\big) \tag{4.1}$$

in the domain $0 < x < \infty$, $0 \le t < T$, with boundary conditions

$$\varphi(0,t) = -\overline{\varphi}, \qquad \varphi(\infty,t) = \pi. \tag{4.2}$$

Time independent solutions of (4.1) satisfy the ODE

$$\varphi'' = f(\varphi), \tag{4.3}$$

where $f(\varphi) = \frac{1}{2}\sin 2\varphi$. This equation has

$$E = \tfrac{1}{2}\varphi'(x)^2 - \tfrac{1}{2}\sin^2\varphi(x)$$

as first integral. It follows that there is exactly one solution $\varphi(E,x)$ of (4.3) which has $\varphi(0) = \frac{1}{2}\pi$ and whose "energy" is E. This solution is determined by the relation

$$-x = \int_{\varphi(E,x)}^{\pi/2} \frac{\mathrm{d}\,\varphi}{\sqrt{2E + \sin^2\varphi}}. \tag{4.4}$$

For $E = 0$ this leads to the unique solution $\Phi(x)$, with $\Phi(-\infty) = 0$, $\Phi(+\infty) = \pi$, and $\Phi(0) = \pi/2$, which corresponds to stereographic projection, namely,

$$\Phi(x) = 2\arctan e^x. \tag{4.5}$$

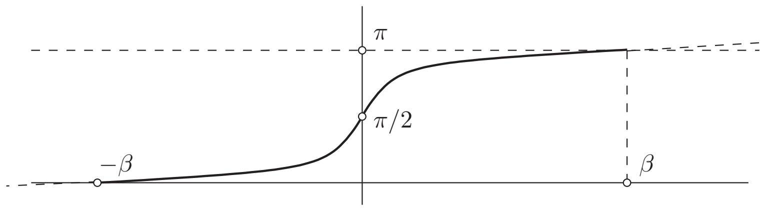

Fig. 4.1. Φ_β.

When $E < 0$, one is led to periodic solutions φ, which we shall not need in this paper. For each $E > 0$ we set

$$\beta(E) = \int_0^{\pi/2} \frac{\mathrm{d}\,\varphi}{\sqrt{2E + \sin^2\varphi}} \tag{4.6}$$

so that the solution with energy E satisfies

$$\varphi(-\beta) = 0, \quad \varphi(0) = \pi/2, \quad \varphi(\beta) = \pi. \tag{4.7}$$

See Figure 4.1.

Clearly, $\beta(E)$ is a monotone function of E with $\beta(E) \to \infty$ as $E \to 0$. We denote the inverse by $E = E_\beta$, and we write $\Phi_\beta(x) = \varphi(E_\beta, x)$. The function Φ_β is thus the unique solution of (4.3) which satisfies the boundary conditions (4.7). One has

$$\lim_{\beta\to\infty} \Phi_\beta(x) = \Phi(x).$$

Lemma 4.1. *The energy E_β of Φ_β satisfies*

$$E_\beta = 8e^{-2\beta+o(1)} \qquad (\beta \to \infty). \tag{4.8}$$

Proof. We have

$$\begin{aligned}\beta &= \int_0^{\pi/2} \frac{d\varphi}{\sqrt{2E + \sin^2\varphi}} \\ &= \int_0^{\pi/2} \frac{1 - \cos\varphi}{\sqrt{2E + \sin^2\varphi}} d\varphi + \int_0^{\pi/2} \frac{\cos\varphi}{\sqrt{2E + \sin^2\varphi}} d\varphi \\ &= A + B.\end{aligned}$$

In the first term we may simply let E tend to 0 because of monotone convergence. One gets

$$\lim_{E\searrow 0} A = \int_0^{\pi/2} \frac{1 - \cos\varphi}{\sin\varphi} d\varphi = \ln 2.$$

For the second term one finds

$$B = \left[\operatorname{arsinh} \frac{\sin\varphi}{\sqrt{2E}}\right]_0^{\pi/2} = \operatorname{arsinh} \frac{1}{\sqrt{2E}}.$$

Adding A and B while using $\operatorname{arsinh} t = \ln\bigl(t + \sqrt{1+t^2}\bigr) = \ln 2t + \frac{1}{4}t^{-2} + \cdots$ for $t \to \infty$, we arrive at

$$\beta = \ln 2 + o(1) + \operatorname{arsinh} \frac{1}{\sqrt{2E}} = \tfrac{1}{2} \ln(8/E) + o(1),$$

from which (4.8) follows. $\square$

LEMMA 4.2. *For all $\beta > 0$ and all $x \geq 0$ one has*

$$0 < \Phi_\beta(x) - \Phi(x) < E_\beta \sinh x \leq Ce^{-2\beta} \sinh x. \tag{4.9}$$

Since $\Phi_\beta - \Phi$ is an odd function, one has the opposite inequalities for $x < 0$.

Proof. From the construction one sees that Φ_β is a monotone function of β. This implies $\Phi_\beta > \Phi$.

Both Φ and Φ_β are solutions of the ODE $\varphi'' - f(\varphi) = 0$, so their difference $z = \Phi_\beta - \Phi$ satisfies $z'' - Q(x)z = 0$, where, by the mean value theorem, one has

$$|Q(x)| = \left| \frac{f(\Phi_\beta(x)) - f(\Phi(x))}{\Phi_\beta(x) - \Phi(x)} \right| < 1.$$

This implies that $z'' - z = (Q(x) - 1)z < 0$. Therefore, taking into account that $z(0) = 0$, one finds

$$\begin{aligned} z(x) &= z(0)\cosh x + z'(0)\sinh x + \int_0^x \sinh(x-\xi)\big\{z''(\xi) - z(\xi)\big\}\, d\xi \\ &< z'(0)\sinh x \end{aligned}$$

for all $x > 0$. Finally,

$$z'(0) = \Phi'_\beta(0) - \Phi'(0) = \frac{\Phi'_\beta(0)^2 - \Phi'(0)^2}{\Phi'_\beta(0) + \Phi'(0)} < E_\beta,$$

because $\Phi'_\beta(0) > \Phi'(0) = 1$. ◻

5. Initial data and their intersections with steady states. Rewritten in the x variable the hypotheses (2.5a) and (2.5b) are

$$\varphi_0(x) \leq \pi \text{ and } \varphi_0'' - f(\varphi_0) < 0 \quad \text{for} \quad 0 \leq x < \infty, \tag{5.1a}$$

$$\lim_{x\to\infty} \varphi_0(x) = \pi. \tag{5.1b}$$

LEMMA 5.1. *Let $\varphi_0 : [0,\infty) \to \mathbb{R}$ be a function which satisfies* (5.1a) *and* (5.1b). *Then $\varphi_0'(x) > 0$ for all $x \geq 0$.*

As a consequence, Lemma 2.1 implies Lemma 2.2.

Proof. There must be a final interval $[x_1, \infty)$ on which $\frac{1}{2}\pi \leq \varphi_0(x) \leq \pi$. On this interval one has $\varphi_0'' < f(\varphi_0) \leq 0$, so φ_0 is concave there. Hence for $x \geq x_1$ we already have $\varphi_0'(x) > 0$.

Let $x_2 \geq 0$ be the largest root of $\varphi'(x) = 0$ if such exists. If we define $E_{\varphi_0}(x) = \frac{1}{2}\big\{\varphi_0'(x)^2 - \sin^2 \varphi_0(x)\big\}$, then for $x > x_2$ we have

$$\frac{d}{dx} E_{\varphi_0}(x) = \varphi_0'(x)\{\varphi_0'' - f(\varphi_0)\} < 0. \tag{5.2}$$

Since $\lim_{x\to\infty} E_{\varphi_0}(x) = 0$, we get $E_{\varphi_0}(x_2) > 0$. On the other hand

$$E_{\varphi_0}(x_2) = \tfrac{1}{2}\big\{\varphi_0'(x_2)^2 - \sin^2 \varphi_0(x_2)\big\} = -\tfrac{1}{2}\sin^2 \varphi_0(x_2) \leq 0,$$

a contradiction.

Hence no such x_2 can exist, and we find that $\varphi_0'(x) > 0$ for all $x \geq 0$. ◻

The proof also shows that $E_{\varphi_0}(x)$ is strictly decreasing for all $x \geq 0$ (by (5.2)) and hence that $E_{\varphi_0}(x) > 0$ for all $x \geq 0$.

Construction of the initial data. It follows from Lemma 5.1 that for any initial φ_0 which satisfies (5.1a), (5.1b) one can invert the map $x \mapsto \varphi_0(x)$ and thus construct a function $\mathcal{G} : [-\bar\varphi, \pi] \to [0, \infty)$ for which one has $\varphi_0'(x) = \mathcal{G}\big(\varphi_0(x)\big)$. This function must satisfy $\mathcal{G}(\pi) = 0$, of course, but also

$$\frac{d}{d\phi}\left(\tfrac{1}{2}\mathcal{G}(\phi)^2 - \tfrac{1}{2}\sin^2\phi\right) < 0 \quad \text{for} \quad -\bar\varphi < \phi < \pi, \tag{5.3}$$

since, by the chain rule and in view of $\varphi_0'(x) = \mathcal{G}(\varphi_0(x))$, the left-hand side equals

$$\frac{\frac{d}{dx}\left(\frac{1}{2}\varphi_0'(x)^2 - \frac{1}{2}\sin^2\varphi_0(x)\right)}{\varphi_0'(x)} = \varphi_0''(x) - f(\varphi_0(x)) < 0 \tag{5.4}$$

when $\phi = \varphi_0(x)$.

Conversely, let $\mathcal{E} : [-\bar\varphi, \pi] \to [0, \infty)$ be any smooth decreasing function for which $\mathcal{E}(\pi) = 0$, and define

$$\mathcal{G}(\phi) = \sqrt{2\big(\mathcal{E}(\phi) + \sin^2\phi\big)}.$$

Then $\mathcal{G}(\phi)$ satisfies (5.3). The solution $\varphi_0 : [0, \infty) \to [-\bar\varphi, \pi)$ of

$$\varphi'(x) = \mathcal{G}(\varphi(x)), \qquad \varphi(0) = -\bar\varphi$$

is an increasing function with $\lim_{x\to\infty} \varphi_0(x) = \pi$. Moreover, (5.4) implies that φ_0 satisfies $\varphi_0'' - f(\varphi_0) < 0$ so that φ_0 satisfies our hypotheses (5.1a) and (5.1b). So, we have constructed an admissible initial value for each smooth decreasing function $\mathcal{E} : [-\bar\varphi, \pi] \to [0, \infty)$ with $\mathcal{E}(\pi) = 0$.

Intersections. We now count intersections of φ_0 with steady states.

LEMMA 5.2. *The graph of $\varphi_0(x)$ intersects the graph of $\Phi(x - \zeta)$ at most once (for any $\zeta \in \mathbb{R}$).*

Proof. All $\Phi(x - \zeta)$ have zero energy, i.e., $E_\Phi \equiv 0$. If $\varphi_0(x_1) = \Phi(x_1 - \zeta)$, then $E_{\varphi_0}(x_1) > E_\Phi(x_1 - \zeta) = 0$ implies $\varphi_0'(x_1) > \Phi'(x_1 - \zeta)$. If there were more than one intersection, at least one of them would have to have $\varphi_0'(x_1) \le \Phi'(x - \zeta)$. ∎

LEMMA 5.3. *The graphs of φ_0 and $\Phi_\beta(x - \zeta)$ intersect at most twice for any $\zeta \in \mathbb{R}$.*

Proof. If some point of intersection $x_1 \ge 0$ has $\varphi_0'(x_1) \le \Phi_\beta'(x_1 - \zeta)$, then one has $E_{\varphi_0}(x_1) \le E_{\Phi_\beta}(x_1)$. But E_{Φ_β} is constant and E_{φ_0} decreases, so for all $x > x_1$ one has $E_{\varphi_0}(x) < E_{\Phi_\beta}(x)$. This implies that there cannot be any further intersections after $x = x_1$, for at such an intersection one would have $\varphi_0'(x) \ge \Phi_\beta'(x - \zeta)$ and thus $E_{\varphi_0}(x) \ge E_{\Phi_\beta}$.

Consequently, there cannot be more than two intersections. For if there were three intersections, say, at $x_1 < x_2 < x_3$, then at either x_1 or x_2 one would have $\varphi_0' \le \Phi_\beta'$, and the third intersection at x_3 could not occur by the argument in the preceding paragraph. ∎

6. Proof of Lemma 2.4 with an error estimate. In section 4 we showed that $\varphi_t < 0$, i.e., $\varphi_{xx} - f(\varphi) < 0$ at each time t. Hence each $\varphi(\cdot, t)$ satisfies the hypotheses of Lemma 2.2, and it follows that $\varphi_x > 0$ for all (x, t). In particular there exists a unique $X(t)$ such that $\varphi(X(t), t) = \frac{1}{2}\pi$. The radius $R(t)$ from (2.6) is given by $R(t) = e^{-X(t)}$.

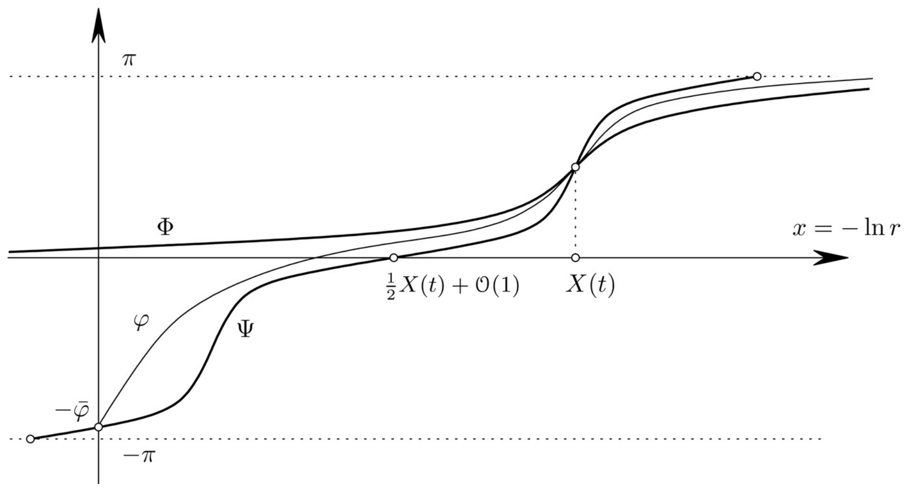

FIG. 6.1. *The proof of Lemma* 2.4.

Again, $\varphi_{xx} - f(\varphi) < 0$ implies that $\varphi(\cdot, t)$ also satisfies Lemma 5.2 and hence the graphs of $\varphi(\cdot, t)$ and $\Phi(x - X(t))$ intersect only once: the intersection occurs by definition of $X(t)$ at $x = X(t)$. We conclude that

$$\begin{cases} \varphi(x,t) > \Phi(x - X(t)) & \text{for} \quad x > X(t), \\ \varphi(x,t) < \Phi(x - X(t)) & \text{for} \quad x < X(t). \end{cases} \tag{6.1}$$

See Figure 6.1. Next, we compare $\varphi(\cdot, t)$ with $\Psi(x)$, where $\Psi(x)$ is the unique solution of

$$\Psi'' - f(\Psi) = 0, \qquad \Psi(X(t)) = \frac{\pi}{2}, \qquad \Psi(0) = -\overline{\varphi}. \tag{6.2}$$

One can extend $\Psi(x)$ to a function on all $\mathbb{R}$ by solving the ODE $\Psi'' = f(\Psi)$.

LEMMA 6.1. $\Psi(x) = \Phi_\beta(x - X(t))$, *where*

$$\beta = \frac{1}{2}X(t) + \mathcal{O}(1). \tag{6.3}$$

Proof. Since $\Psi(x)$ crosses the $\varphi = 0$ line, it must be a positive energy solution of the ODE and hence $\Psi(x) = \Phi_\beta(x - X(t))$ for some β. At $x = 0$ one has $\Psi(0) = -\overline{\varphi}$, which is bounded away from 0 and $-\pi$, so within a distance[1] of $\mathcal{O}(1)$ there must be a point x_1 with $\Psi(x_1) = -\frac{1}{2}\pi$. Clearly, $X(t) - x_1 = 2\beta$. ∎

LEMMA 6.2. *For all* $x \geq 0$ *one has*

$$|\varphi(x,t) - \Phi(x - X(t))| \leq Ce^{-X(t)}\big|\sinh(x - X(t))\big|. \tag{6.4}$$

Proof. By Lemma 5.3 the graphs of $\varphi(x,t)$ and $\Psi(x)$ intersect at most twice, which they do at $x = 0$ and at $x = X(t)$. Hence we have

$$\begin{cases} \varphi(x,t) < \Psi(x) & \text{for } x > X(t) \\ \varphi(x,t) > \Psi(x) & \text{for } x < X(t). \end{cases} \tag{6.5}$$

[1]In fact, using $E_\beta \geq 0$ one can estimate x_1 by $|x_1| = |\int_{\overline{\varphi}}^{\pi/2}(2E_\beta + \sin^2\varphi)^{-1/2}d\varphi| \leq |\text{artanh } \cos\overline{\varphi}|$.

If we combine this with (6.1) we get

$$|\varphi(x,t) - \Phi(x - X(t))| \le |\Psi(x) - \Phi(x - X(t))|$$

for all $x \ge 0$. Since $\Psi(x) = \Phi_\beta(x - X(t))$, the inequality (6.4) follows from Lemma 4.2. ∎

7. Convergence of higher derivatives. We are considering φ as a function of $x = -\ln r$ and t so that φ satisfies (4.1), (4.2). In these variables blow-up of the harmonic map flow leads to unbounded time derivatives but not to unbounded space derivatives.

LEMMA 7.1. *For all* $m = 0, 1, 2, \ldots,$ *there are constants* $M_{m,\delta}$ *such that*

$$\left|\frac{\partial^m \varphi}{\partial x^m}(x,t)\right| \le M_{m,\delta} \tag{7.1}$$

holds for $x \ge 1$ *and* $t \ge \delta$.

Proof. Let $x_0 \ge 1$ and $t_0 \in [\delta, T)$ be given. Then consider

$$\widetilde{\varphi}(x,t) = \varphi(x_0 + x, t_0 + e^{-2x_0} t).$$

The function $\widetilde{\varphi}$ satisfies (4.1) on the rectangle $-1 < x < 1$, $-\delta < t \le 0$ (in fact for $-\delta e^{2x_0} < t \le 0$) and is bounded there.

By standard interior estimates for semilinear parabolic equations, we now find that all derivatives $\partial_x^m \widetilde{\varphi}(0,0)$ are bounded. This implies (7.1). ∎

Recall that for any integers $0 \le k \le m$ there is a constant $C_{k,m}$ such that any C^m function on the interval $[-L, L]$ satisfies

$$\|f^{(k)}\|_\infty \le C_{k,m} \|f\|_\infty^{1-k/m} \|f^{(m)}\|_\infty^{k/m},$$

$\|\cdots\|_\infty$ being the supremum norm on the interval $[-L, L]$. If we apply this interpolation inequality to (6.4) and (7.1), we find that $\varphi(X(t) + z, t)$ converges in C^∞ to $\Phi(z)$. More precisely, we get the following estimates.

LEMMA 7.2. *For any* $0 \le k \le m$ *there is a constant* $C_{k,m,L}$ *such that*

$$\sup_{|x - X(t)| \le L} \left|\frac{\partial^k \varphi}{\partial x^k} - \Phi^{(k)}\right| \le C_{k,m,L} e^{-(1-k/m)X(t)}. \tag{7.2}$$

8. The mollified logarithmic radius $\hat{X}$. We consider the following alternative to $X(t)$: define $\hat{X}(t)$ by requiring

$$\int_{-1}^{+1} \eta(s) \varphi(\hat{X}(t) + s, t) ds = \frac{\pi}{2},$$

where $0 \le \eta \in C_c^\infty(-1, 1)$ is some function with $\int_{-1}^1 \eta(s) ds = 1$. The left-hand side here is strictly increasing as a function of $\hat{X}(t)$ so that uniqueness of $\hat{X}(t)$ is ensured.

We also define the corresponding mollified radius

$$\hat{R}(t) = e^{-\hat{X}(t)}.$$

LEMMA 8.1. $|\hat{X}(t) - X(t)| \le C e^{-X(t)}$ *for some* $C < \infty$.

Consequently, one also has

$$X(t) = \hat{X}(t) + o(1) \text{ and } R(t) = (1 + o(1)) \hat{R}(t) \text{ as } t \nearrow T. \tag{8.1}$$

Proof. It follows from Lemma 6.2 that

$$\int_{-1}^{+1} \eta(s)\varphi(X(t)+s,t)ds = \frac{\pi}{2} + \mathcal{O}(e^{-X(t)}).$$

It also follows from Lemma 7.2 that

$$A(\xi) \stackrel{\text{def}}{=} \int_{-1}^{+1} \eta(s)\varphi(X(t)+\xi+s,t)ds$$

satisfies

$$A'(\xi) = \int_{-1}^{+1} \eta(s)\varphi_x(X(t)+\xi+s,t)ds \geq \delta > 0$$

for some constant δ, and all $|\xi| \leq 1$.

Writing $\hat{X}(t) = X(t) + \xi$, these two inequalities imply the lemma. ∎

We proceed to compute $\hat{X}'(t)$. Differentiation of the defining relation for $\hat{X}(t)$ gives

$$0 = \frac{d}{dt}\int_{-1}^{1} \eta(s)\varphi(\hat{X}(t)+s,t)dt = \int_{-1}^{1} \eta(s)\{\varphi_t + \hat{X}'(t)\varphi_x\}ds$$

so that

$$\hat{X}'(t) = -\frac{\int_{-1}^{1} \eta(s)\varphi_t(\hat{X}(t)+s,t)ds}{\int_{-1}^{1} \eta(s)\varphi_x(\hat{X}(t)+s,t)ds}.$$

It is immediately clear from $\varphi_t < 0$ and $\varphi_x > 0$ that

$$\hat{X}'(t) > 0. \tag{8.2}$$

Moreover, the PDE (4.1) for φ implies

$$\hat{X}'(t) = -\frac{\int_{-1}^{1} \eta(s)e^{2(\hat{X}+s)}(\varphi_{ss} - \frac{1}{2}\sin 2\varphi)ds}{\int_{-1}^{1} \eta(s)\varphi_s(\hat{X}+s,t)ds}.$$

(Note that for fixed t one has $\partial/\partial x = \partial/\partial s$.) After factoring out the $e^{2\hat{X}}$ and integrating by parts twice in the numerator and once in the denominator, one gets

$$\hat{X}'(t) = e^{2\hat{X}(t)}\frac{\int_{-1}^{1} \{(\eta(s)e^{2s})_{ss}\varphi - \eta(s)e^{2s}\frac{1}{2}\sin 2\varphi\}ds}{\int_{-1}^{1} \eta'(s)\varphi(\hat{X}+s,t)ds}. \tag{8.3}$$

By Lemma 6.2, we find for the denominator

$$\begin{aligned}\int_{-1}^{1} \eta'(s)\varphi(\hat{X}+s,t)ds &= \int_{-1}^{1} \eta'(s)\Phi(s)ds + \mathcal{O}(e^{-\hat{X}(t)})\\ &= -\int_{-1}^{1} \eta(s)\Phi'(s)ds + \mathcal{O}(e^{-\hat{X}(t)})\\ &= -C_0 + \mathcal{O}(e^{-\hat{X}(t)}),\end{aligned}$$

in which C_0 is some positive constant.

For the numerator we get, using Lemma 6.2 again,

$$\begin{aligned}
&\int_{-1}^{1}\left\{(\eta(s)e^{2s})_{ss}\varphi - \eta(s)e^{2s}\tfrac{1}{2}\sin 2\varphi\right\}ds \\
&\quad = \int_{-1}^{1}\left\{(\eta(s)e^{2s})_{ss}\Phi(s) - \eta(s)e^{2s}\tfrac{1}{2}\sin 2\Phi(s)\right\}ds + \mathcal{O}(e^{-\hat{X}(t)}) \\
&\quad = \int_{-1}^{1}\eta(s)e^{2s}\left\{\Phi''(s) - \tfrac{1}{2}\sin 2\Phi(s)\right\}ds + \mathcal{O}(e^{-\hat{X}(t)}) \\
&\quad = \mathcal{O}(e^{-\hat{X}(t)}),
\end{aligned}$$

in which we have used that Φ satisfies the differential equation $\Phi''(s) = \frac{1}{2}\sin 2\Phi(s)$.

These last two computations applied to (8.3) give us

$$\hat{X}'(t) = \mathcal{O}(e^{\hat{X}(t)}), \tag{8.4}$$

which is the main estimate we derive in this section. Since $\hat{R}(t) = e^{-\hat{X}(t)}$, we have

$$\frac{d\hat{R}}{dt} = -e^{-\hat{X}(t)}\hat{X}'(t) = \mathcal{O}(1),$$

which implies $\hat{R}(t) = \mathcal{O}(T-t)$ and, by Lemma 8.1,

$$R(t) = \mathcal{O}(T-t). \tag{8.5}$$

9. $\boldsymbol{m(\tau)}$ and $\boldsymbol{Y(\tau)}$. We consider the parabolic blow-up of our solution to harmonic map flow. Let

$$\varphi(r,t) = u\left(\frac{r}{\sqrt{T-t}}, -\ln(T-t)\right),$$
$$y = \frac{r}{\sqrt{T-t}}, \qquad \tau = -\ln(T-t).$$

Then $u(y,\tau)$ is defined for $0 \le y \le e^{\tau/2}$, $-\ln T \le \tau < \infty$, where it satisfies

$$u_\tau = u_{yy} + \left(\frac{1}{y} - \frac{y}{2}\right)u_y - \frac{1}{y^2}u + \frac{g(u)}{y^2}, \tag{9.1a}$$

$$= \frac{1}{y}e^{y^2/4}\left(ye^{-y^2/4}u_y\right)_y - \frac{1}{y^2}u + \frac{g(u)}{y^2}, \tag{9.1b}$$

in which

$$g(u) \stackrel{\text{def}}{=} u - \tfrac{1}{2}\sin 2u = \tfrac{2}{3}u^3 + \mathcal{O}(u^5).$$

We define

$$Y(\tau) = e^{\tau/2}R(T - e^{-\tau}).$$

Then we have shown that

$$Y(\tau) \le Ce^{-\tau/2} \tag{9.2}$$

for some constant $C < \infty$.

LEMMA 9.1. *If one defines $U(\eta) = \pi - 2\arctan\eta$, then for all $y \geq Y(\tau)$ one has*

$$\left|u(y,\tau) - U\left(\frac{y}{Y(\tau)}\right)\right| \leq \begin{cases} Ce^{-\tau/2}y & \text{for } y \geq Y(\tau), \\ Ce^{-\tau/2} & \text{for } 0 \leq y \leq Y(\tau). \end{cases} \tag{9.3}$$

Proof. Lemma 6.2 implies that

$$|u(y,\tau) - U(y/Y(\tau))| \leq Ce^{-X(\tau)}\left|\frac{y}{Y} - \frac{Y}{y}\right| \leq Ce^{-\tau/2}Y\left|\frac{y}{Y} - \frac{Y}{y}\right|.$$

In the region $y \geq Y$ this directly implies the first inequality in (9.3).

For $0 \leq y \leq Y$ we get

$$|u - U| \leq Ce^{-\tau/2}Y^2/y.$$

In this region we also have $U \leq u \leq \pi$ so that

$$|u - U| \leq \pi - U = 2\arctan y/Y \leq 2y/Y.$$

At each $y \in [0, Y]$ these two estimates imply that

$$|u(y,\tau) - U(y/Y(\tau))| \leq \min\{2y/Y, Ce^{-\tau/2}Y^2/y\} \leq C\sqrt{Y}e^{-\tau/4} \leq Ce^{-\tau/2},$$

since $\min\{a,b\} \leq \sqrt{ab}$ and in view of the estimate (9.2) for $Y(\tau)$. ◻

We define

$$m(\tau) = \int_0^{e^{\tau/2}} y^2 e^{-y^2/4} u(y,\tau)\, dy. \tag{9.4}$$

LEMMA 9.2. *One has*

$$m'(\tau) = -\tfrac{1}{2}m(\tau) + (4 + o(1))Y(\tau) + \mathcal{O}(e^{-\frac{3}{2}\tau}). \tag{9.5}$$

Proof. One differentiates the defining equation (9.4) for $m(\tau)$ and obtains

$$\begin{aligned} m'(\tau) &= e^{\tau}e^{-e^{\tau}/4}u(e^{\tau/2},\tau) + \int_0^{e^{\tau/2}} y^2 e^{-y^2/4}u_\tau(y,\tau)\, dy \\ &= \epsilon(\tau) + \int_0^{e^{\tau/2}} \left\{y(ye^{-y^2/4}u_y)_y - e^{-y^2/4}u + e^{-y^2/4}g(u)\right\} dy \end{aligned}$$

(integrate by parts twice)

$$\begin{aligned} &= \epsilon(\tau) - \tfrac{1}{2}\int_0^{e^{\tau/2}} y^2 e^{-y^2/4}u(y,\tau)\, d\tau + \int_0^{e^{\tau/2}} e^{-y^2/4}g(u)dy \\ &= -\tfrac{1}{2}m(\tau) + \int_0^{e^{\tau/2}} e^{-y^2/4}g(u)dy + \epsilon(\tau). \end{aligned}$$

Here $\epsilon(\tau)$ stands for a function of time which vanishes super exponentially, i.e., for some $C, c > 0$ one has

$$|\epsilon(\tau)| \leq Ce^{-ce^{\tau}}.$$

To complete the proof we must estimate the remaining integral. This is done in the following two propositions. □

PROPOSITION 9.3. $\int_0^\infty g(U(y))dy = 4$.

Proof. Since $U(y)$ satisfies the ODE $U'' + \frac{1}{y}U' = \frac{1}{2}y^{-2}\sin 2U$, we have after integrating by parts a few times

$$\begin{aligned}\int_0^L g(U(y))dy &= \int_0^L \left(-y^2U''(y) - yU'(y) + U(y)\right)\, dy\\ &= [-y^2U'(y) + yU(y)]_{y=0}^{y=L}\\ &= -L^2U'(L) + LU(L).\end{aligned}$$

For large y we have $U(y) \sim 2/y$ and hence $U'(y) \sim -2/y^2$. Taking the limit $L \to \infty$ in the above computation then proves the lemma. □

PROPOSITION 9.4.

$$\int_0^{e^{\tau/2}} g(u(y,\tau))e^{-y^2/4}\, dy = 4Y(\tau) + o(Y(\tau)) + \mathcal{O}(e^{-(3/2)\tau}).$$

Proof. We split the integral into several pieces.

$$\begin{aligned}\int_0^{e^{\tau/2}} g(u(y,\tau))e^{-y^2/4}\, dy &= \int_0^{e^{\tau/2}} g(U(y/Y))e^{-y^2/4}\, dy\\ &\qquad + \int_0^{e^{\tau/2}} \{g(u(y,\tau)) - g(U(y/Y))\}e^{-y^2/4}\, dy\\ &= I_1 + I_2.\end{aligned}$$

In the first integral we substitute $y = \eta Y$. The variable η then runs from 0 to $e^{\tau/2}/Y(\tau) \geq ce^{\tau}$. One finds

$$\begin{aligned}I_1 &= Y(\tau)\int_0^{e^{\tau/2}/Y(\tau)} g(U(\eta))e^{-\eta^2Y^2/4}\, d\eta\\ &= Y(\tau)(1+o(1))\int_0^\infty g(U(\eta))e^{-\eta^2Y^2/4}\, d\eta\\ &= Y(\tau)(1+o(1))\int_0^\infty g(U(\eta))\, d\eta\\ &= (4+o(1))Y(\tau)\end{aligned}$$

by Lemma 9.3 and monotone convergence.

In the second integral we use the mean value theorem, i.e., $g(u)-g(U) = g'(\tilde{u})(u-U)$ for some $\tilde{u}$ between u and U. Furthermore,

$$0 \leq g'(u) = 1 - \cos 2u \leq Cu^2$$

for some constant, so we get $0 \leq g'(\tilde{u}) \leq C(U^2 + u^2)$. On the interval $Y \leq y \leq e^{\tau/2}$ Lemma 9.1 tells us that

$$U(y/Y(\tau)) - Ce^{-\tau/2}y \leq u(y,\tau) \leq U(y/Y(\tau))$$

so that $u^2 \le U^2 + Ce^{-\tau} y^2$. Applying this to the integral I_2 on the interval $Y \le y \le e^{\tau/2}$ we get

$$\left| \int_Y^{e^{\tau/2}} \{g(u(y,\tau)) - g(U(y/Y))\} e^{-y^2/4}\, dy \right| \le$$

$$\begin{aligned} &\le C \int_Y^{e^{\tau/2}} (U^2 + u^2)|U - u| e^{-y^2/4}\, dy \\ &\le Ce^{-\tau/2} \int_Y^{e^{\tau/2}} \left\{ \left(\frac{Y}{y}\right)^2 + e^{-\tau} y^2 \right\} y e^{-y^2/4}\, dy \\ &\le Ce^{-\tau/2} Y(\tau)^2 \ln \frac{1}{Y(\tau)} + Ce^{-(3/2)\tau} \\ &\le o(Y(\tau)) + Ce^{-(3/2)\tau}. \end{aligned}$$

For the integral from $y = 0$ to $y = Y$ we have $|u - U| = \mathcal{O}(e^{-\tau/2}) = o(1)$ uniformly, by Lemma 9.1. Since $g'(u)$ is bounded, this implies

$$\int_0^{Y(\tau)} |g(u(y,\tau)) - g(U(y/Y))| e^{-y^2/4}\, dy = o(Y(\tau)).$$

Adding the two pieces we get

$$|I_2| \le \left\{ Ce^{-\frac{3}{2}\tau} + o(Y(\tau)) \right\}.$$

The lemma is proved by adding the estimates for I_1 and I_2. ▯

We can now improve our bound for the blow-up rates of $X(t)$ and $Y(\tau)$.

PROPOSITION 9.5. *For some constant $C < \infty$ one has $|m(\tau)| \ge Ce^{-\tau/2}$.*

Proof. It follows from the estimates in Lemma 9.1 that $u(y,\tau) \ge -Cye^{-\tau/2}$ for all y and τ. Substitution in (9.4) then gives $m(\tau) \ge -Ce^{-\tau/2}$.

To get the opposite inequality we recall Lemma 9.3, which says that for $y \ge Y$ one has $u(y,\tau) \le U(y/Y)$, while for $y \in [0, Y]$ one has $u(y,\tau) \le U(y/Y) + Ce^{-\tau/2}$. The explicit expression for $U(y/Y)$ implies that $U(y/Y) \le 2Y/y$. Substitution of these estimates in the definition of $m(\tau)$ gives $m(\tau) \le Ce^{-\tau/2}$. ▯

PROPOSITION 9.6.

$$\int_{\tau_0}^{\infty} e^{\tau/2} Y(\tau)\, d\tau < \infty. \tag{9.6}$$

Proof. Apply the variation of constants formula to (9.5) to get

$$e^{\tau/2} m(\tau) - e^{\tau_0/2} m(\tau_0) = \int_{\tau_0}^{\tau} \left\{ (4 + o(1)) e^{\sigma/2} Y(\sigma) + \mathcal{O}(e^{-\sigma}) \right\} d\sigma.$$

Since $Y(\tau) > 0$ and since the left-hand side is bounded from above by the previous proposition, (9.6) follows. ▯

Unraveling the definitions of Y and τ, we find

$$\int_0^T R(t) \frac{dt}{(T-t)^2} < \infty. \tag{9.7}$$

To conclude we show how this integral bound also implies a pointwise bound. Recall that $R(t)$ is monotone, so that for any τ_0 and $\tau \in (\tau_0 - 1, \tau_0]$ it follows from $Y(\tau) = e^{\tau/2}R(T - e^{-\tau})$ that

$$Y(\tau) > e^{-1/2}Y(\tau_0).$$

Hence

$$\int_{\tau_0-1}^{\tau_0} e^{\tau/2}Y(\tau)\, d\tau > e^{\tau_0/2}e^{-1}Y(\tau_0).$$

Convergence of the integral (9.6) then implies

$$\lim_{\tau_0\to\infty} e^{\tau_0/2}Y(\tau_0) = 0,$$

which in turn implies $R(t) = o(T - t)$ as $t \nearrow T$.

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