

Self-intersecting Geodesics and Entropy of the Geodesic Flow

Sigurd Angenent

This note was inspired by a paper of Denvir and MacKay [1] who showed (among other things) that the geodesic flow on a torus \mathbb{T}^2 with Riemannian metric g has positive topological entropy as soon as it admits a contractible closed geodesic.

Our main observation concerns closed geodesics on surfaces M with a smooth Finsler metric, i.e. a function $F : TM \rightarrow [0, \infty)$ which is a norm on each tangent space T_pM , $p \in M$, which is smooth outside of the zero section in TM , and which is strictly convex in the sense that $\text{Hess}(F^2)$ is positive definite on $T_pM \setminus \{0\}$.

One calls a Finsler metric F *symmetric* if $F(p, -v) = F(p, v)$ for all $v \in T_pM$. We denote the universal cover of a surface M by \hat{M} . Any Finsler metric on M lifts to a Finsler metric on \hat{M} which we again denote by F .

LEMMA. *Let M be a compact surface with $\chi(M) \leq 0$, and let $F : TM \rightarrow \mathbb{R}$ be a smooth symmetric Finsler metric on M . If the lift $\hat{\gamma} : \mathbb{R} \rightarrow \hat{M}$ of some closed geodesic $\gamma : \mathbb{R} \rightarrow M$ has a self intersection, then (\hat{M}, F) admits a simple closed geodesic whose projection to M is thus a contractible closed geodesic for (M, F) .*

Since the Denvir-MacKay result generalizes to the case of Finsler metrics (see § 3) we get:

COROLLARY. *The lift $\hat{\gamma} : \mathbb{R} \rightarrow \mathbb{R}^2$ of any closed geodesic on a Finsler torus (\mathbb{T}^2, F) whose geodesic flow has zero topological entropy must be an embedded curve.*

1. Proof of the Lemma

Choice of a background metric. Since $\chi(M) \leq 0$ the universal cover \hat{M} can be equipped with a standard Riemannian metric of constant curvature. We will assume that \hat{M} is either the Euclidean plane, in case $M = \mathbb{T}^2$, or the Poincaré disc when $\chi(M) < 0$. In either case there is a natural distance function between points $p, q \in \hat{M}$, which we denote by $d(p, q)$. The open disc with center p and radius $r > 0$ will be denoted by $D_r(p)$.

Partially supported by NSF grant DMS-0101124, and NWO through a visitor's fellowship B 61-581.

Since M is compact there is a constant C such that for any differentiable arc α in \hat{M} the length $\ell(\alpha)$ measured with the Finsler metric and the length $L(\alpha)$ measured with the standard constant curvature metric satisfy

$$C^{-1}\ell(\alpha) \leq L(\alpha) \leq C\ell(\alpha).$$

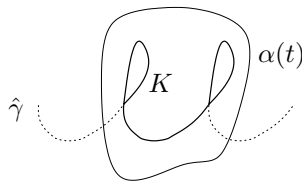
Construction of a simple closed geodesic. Let $\gamma : \mathbb{R} \rightarrow M$ be a closed non-contractible geodesic, so that $\gamma(x+1) = \gamma(x)$ for all $x \in \mathbb{R}$. For some nontrivial decktransformation $f : \hat{M} \rightarrow \hat{M}$ one then has $\hat{\gamma}(x+1) = f(\hat{\gamma}(x))$ for all $x \in \mathbb{R}$. Since $\hat{\gamma}$ has a self intersection, there exist $a < b$ such that $\hat{\gamma}(a) = \hat{\gamma}(b)$. Define

$$K = \hat{\gamma}([a, b+1]).$$

Let $\mathcal{O} \subset \hat{M}$ be the unbounded component of $\hat{M} \setminus K$. For any $p \in K$ and sufficiently large $r > 0$ one has $K \subset D_r(p)$. The fundamental group of \mathcal{O} is generated by $\partial D_r(p)$.

Let $\{\alpha(t) : 0 \leq t < T\}$ be the maximal solution to Finsler curve shortening (as described in § 2) on (\hat{M}, g) with $\alpha_0 = \partial D_r(p)$ as initial value. The initial curve $\alpha(0)$ is disjoint from K , and by the maximum principle this condition persists (this is where symmetry of the Finsler metric is used). Indeed, if for some $\bar{t} \in (0, T)$ one had $\alpha(\bar{t}) \cap K \neq \emptyset$, then there would be a first time $t_1 \in (0, T)$ at which $\alpha(t_1)$ and K meet. Thus there is some $x_1 \in [a, b+1]$ for which $\hat{\gamma}(x_1)$ lies on $\alpha(t_1)$. If $x_1 = a$ or $x_1 = b+1$ then, since $\hat{\gamma}(a) = \hat{\gamma}(b)$ and $\hat{\gamma}(a+1) = \hat{\gamma}(b+1)$, we can replace x_1 with b or $a+1$. We can therefore assume that $x_1 \in (a, b+1)$. But then the first point of contact between $\alpha(t_1)$ and $\hat{\gamma}$ must be a tangency, which is excluded by the maximum principle.

Let $L = \ell(\alpha(0))$ be the length of the initial curve. The length of any subsequent $\alpha(t)$ is no more than L . Hence the length of $\alpha(t)$ measured with the background metric will never exceed CL . If at some time $t \in [0, T)$ the curve $\alpha(t)$ were to contain a point q with $d(p, q) > r + CL$ then $\alpha(t) \subset D_{CL/2}(q)$. This disc is disjoint from $D_r(p)$, and hence from K , so that $\alpha(t) \subset D_{CL/2}(q)$ would imply that $\alpha(t)$ and hence $\alpha(0)$ is contractible in \mathcal{O} . This cannot be, so we conclude that the entire evolving family of curves $\{\alpha(t) : 0 \leq t < T\}$ is contained in $D_{r+CL}(p)$.



Since the curves $\alpha(t)$ cannot escape to infinity or shrink to a point, the Grayson-Oaks theorem, which we describe below, tells us that $T = \infty$ and that the $\alpha(t)$ must accumulate to simple closed geodesics, i.e. any sequence $t_i \nearrow \infty$ has a subsequence t_{i_j} such that $\alpha(t_{i_j})$ converges smoothly to some simple closed geodesic. Hence such a geodesic must exist.

2. Finsler curve shortening

If g is Riemannian metric then a family of curves $\alpha(t)$ evolves by curve shortening if one has $V = \kappa$, where V is the normal component of the velocity of $\alpha(t)$, and κ is its geodesic curvature. Grayson showed

THEOREM. (Grayson, [2]) *Let $\{\alpha(t) : 0 \leq t < T\}$ be a maximal smooth solution which remains in a compact subset of \hat{M} . If the initial curve is embedded then one either has $T < \infty$, in which case $\alpha(t)$ shrinks to a point as $t \nearrow T$, or else $T = \infty$, and in this case the $\alpha(t)$ accumulate on simple closed geodesics as $t \nearrow \infty$.*

If F is a symmetric Finsler metric, then we define F -curve shortening as follows. Identify \hat{M} with either the Euclidean plane or the Poincaré disc as above, and measure the normal velocity and geodesic curvature with respect to this background metric. For any evolving family of curves $\alpha : \mathbb{R} \times [0, T) \rightarrow \mathbb{R}^2$ denote the unit tangent and normal by \mathcal{T} and \mathcal{N} . Assume the $\alpha(t)$ are parameterized so that $\alpha_t \perp \alpha_s$ in the background metric. We can then write $\alpha_t = v\mathcal{N}$ with v the normal velocity. The first variation of the Finsler length ℓ of $\alpha(t)$ is

$$\frac{d\ell(\alpha(t))}{dt} = \frac{d}{dt} \int F(\alpha, \alpha_s) ds = \int \{F_{\alpha^i} v \mathcal{N}^i + F_{\alpha_s^i} (\alpha_s^i)_t - v \kappa F\} ds.$$

The commutation relation $[\partial_t, \partial_s] = \kappa v \partial_s$ implies

$$(\alpha_s^i)_t = (\alpha_t^i)_s + \kappa v \alpha_s^i = v_s \mathcal{N}^i + v \mathcal{N}_s^i + \kappa v \mathcal{T}^i = v_s \mathcal{N}^i$$

since $\mathcal{N}_s = -\kappa \mathcal{T}$. The Finsler norm is homogeneous of degree one on each tangent space $T_p \hat{M}$, so that Euler's identity implies $\alpha_s^i F_{\alpha_s^i} = F$. After substituting these identities and integrating by parts we get

$$\begin{aligned} \frac{d\ell(\alpha(t))}{dt} &= \int \{F_{\alpha^i} \mathcal{N}^i - (F_{\alpha_s^i} \mathcal{N}^i)_s - \kappa F\} v ds \\ &= \int \{F_{\alpha^i} \mathcal{N}^i - F_{\alpha_s^i \alpha^j} \mathcal{N}^i \mathcal{T}^j - F_{\alpha_s^i \alpha_s^j} \mathcal{N}^i \mathcal{T}_s^j - F_{\alpha_s^i} \mathcal{N}_s^i - \kappa F\} v ds \\ &= \int \{F_{\alpha^i} \mathcal{N}^i - F_{\alpha_s^i \alpha^j} \mathcal{N}^i \mathcal{T}^j - F_{\alpha_s^i \alpha_s^j} \mathcal{N}^i \mathcal{N}^j \kappa\} v ds \end{aligned}$$

We now declare that the curves $\alpha(t)$ evolve by curve shortening on \hat{M} (with background metric g) if they satisfy

$$(1) \quad v = F_{\alpha_s^i \alpha_s^j} \mathcal{N}^i \mathcal{N}^j \kappa - F_{\alpha_s^i \alpha^j} \mathcal{N}^i \mathcal{T}^j.$$

Since F is a strictly convex Finsler metric the quantity $F_{\alpha_s^i \alpha_s^j} \mathcal{N}^i \mathcal{N}^j$ is uniformly bounded from above and below by positive constants. Therefore we have just defined curve shortening in terms of a quasilinear parabolic equation for curves to which the results of Oaks [3] apply. One has

THEOREM. (Oaks, [3]) *Grayson's theorem continues to hold if Riemannian curve shortening is replaced by Finsler curve shortening (1) for a symmetric Finsler metric.*

We also see from equation (1) that geodesics of the Finsler metric F are smooth curves which satisfy $v = 0$, i.e.

$$(2) \quad \kappa = \frac{F_{\alpha_s^i \alpha^j} \mathcal{N}^i \mathcal{T}^j}{F_{\alpha_s^i \alpha_s^j} \mathcal{N}^i \mathcal{N}^j}.$$

3. The Denvir-MacKay theorem for Finsler metrics on \mathbb{T}^2

Let $\hat{\gamma}$ be a closed geodesic in \mathbb{R}^2 for a Finsler metric $F : T\mathbb{R}^2 \rightarrow [0, \infty)$ which comes from lifting a Finsler metric on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Since the image of $\hat{\gamma}$ is compact a large enough $n \in \mathbb{N}$ exists for which $\hat{\gamma}$ and all its translates by vectors $n\mathbf{v}$ with $\mathbf{v} \in \mathbb{Z}^2$ are disjoint.

Let \mathcal{U} be the set of points in the plane which lie outside to each translate $\hat{\gamma} + n\mathbf{v}$, with $\mathbf{v} \in \mathbb{Z}^2$. The homotopyclass of any closed curve α in \mathcal{U} contains a curve which minimizes the Finsler length $\ell(\alpha)$. Since $\partial\mathcal{U}$ consists of geodesics the minimizer cannot be tangent to $\partial\mathcal{U}$ unless α is one of the translates $\hat{\gamma} + n\mathbf{v}$. It follows that the minimizer is a smooth geodesic satisfying (2). The abundance of closed geodesics exhibited by this construction implies the geodesic flow has positive topological entropy.

References

1. J. Denvir and R. S. MacKay, *Consequences of contractible geodesics on surfaces*, Trans. Amer. Math. Soc. **350** (1998), no. 11, 4553–4568. MR [MR1615951](#) ([99b:58183](#))
2. Matthew A. Grayson, *Shortening embedded curves*, Ann. of Math. (2) **129** (1989), no. 1, 71–111. MR [MR979601](#) ([90a:53050](#))
3. Jeffrey Oaks, *Singularities and self-intersections of curves evolving on surfaces*, Indiana Univ. Math. J. **43** (1994), 959–981.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN AT MADISON, ON LEAVE AT THE VRIJE UNIVERSITEIT AMSTERDAM