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THE TOPOLOGICAL ENTROPY AND INVARIANT CIRCLES OF AN AREA PRESERVING TWISTMAP.

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Let A be the annulus $S^1 \times [0, 1]$, and let $f : A \to A$ be an area preserving twist homeomorphism of A. The two boundary components of A, $A_j = S^1 \times \{j\}$, are invariant under f^q, and we shall denote the rotation number of $f|A_j$ by ρ_j.

In this note we wish to point out that the folowing holds:

Theorem A. *If the topological entropy $h_{top}(f)$ of f vanishes, then f must have an invariant circle of rotation number ω, for any $\omega \in (\rho_0, \rho_1)$.*

In fact, we'll show that if "one of the invariant circles of f is missing," there must exist a compact subset $K \subset A$ which is invariant under f^q, for some $q \geq 1$, and such that $f^q|K$ has a Bernoulli shift as a factor.

If the map f is a $C^{1,\epsilon}$ diffeomorphism, then a theorem of A. Katok implies that f must have a "horse shoe" if $h_{\text{top}}(f) > 0$. Thus our theorem says that *any $C^{1,\epsilon}$ twist diffeomorphism of the annulus either has a transversal homoclinic point, or else it has invariant circles for any rotation number in its rotation interval* (ρ_0, ρ_1).

We shall give two proofs of this theorem. The first proof consists of simply combining two results obtained by Dick Hall and Phil Boyland.

Indeed, in [3] they showed that, if one of the invariant circles of f is missing, then for some p and q with $\gcd(p, q) = 1$ the map must have a periodic orbit of type (p, q) which is not a Birkhoff orbit. On the other hand, Boyland showed in [2] that a twistmap with a non Birkhoff periodic orbit of type (p, q) $(\gcd(p, q) = 1$ must have positive topological entropy, which clearly implies the theorem.

Boyland's proof of the second result which we just quoted is a fine application of Thurston's classification of surface diffeomorphisms: one of the points we wish to make in this note is that one can give an "elementary" proof of theorem A. Indeed, in [1] we gave an alternative and self contained proof of Boyland's criterion for positive entropy. We'll show that the method of [1] can also be used to prove theorem A.

We begin the proof by recalling some of the conclusions of Birkhoff's investigations on twistmaps. Birkhoff showed that any invariant circle is the graph of a Lipschitz continuous function (i.e. has the form $\{(x, \varphi(x)) : x \in S^1\}$) and that the Lipschitz constant of φ is bounded by a constant which only depends on the map f. To be sure, Birkhoff only proved this for C^1 maps, but as A. Katok observed in [4] the result is also true for twist homeomorphisms, if one replaces "Lipschitz continuous" by "continuous," and "Lipschitz constant" by "modulus of continuity." The use of this estimate is that it implies that the set of invariant circles (and hence the set of rotation numbers which can occur) is closed.

If we assume that at least one invariant circle is missing, it follows that there is an entire interval (ρ_1, ρ_2) such that the map will not have a invariant circle with rotation number ρ for any $\rho \in (\rho_1, \rho_2)$. We may assume that (ρ_1, ρ_2) is a maximal interval with this property, so that there will be invariant circles $y = \varphi_k(x)$ with rotation numbers ρ_k $(k = 1, 2)$, and none for $\rho_1 < \rho < \rho_2$. The region

$$Z = \{(x, y) \mid x \in S^1, \varphi_1(x) \le y \le \varphi_2(x)\}$$

is what Birkhoff called a *zone of instability*. This region is homeomorphic to an annulus, e.g. via the homeomorphism $\Phi : A \to Z$ given by

$$\Phi(x, y) = (x, y\varphi_1(x) + (1 - y)\varphi_2(x)).$$

If we conjugate the twistmap f with this homeomorphism, then the result, $g = \Phi^{-1} f \Phi$, is again a twistmap, as the reader can easily verify.

We shall now forget about the original twistmap, and continue with g, which, by Birkhoff's construction, has no invariant circles at all, except the two boundary components of the annulus A.

To construct special orbits of the new map $g : A \to A$, we'll consider its generating function $h(x_0, x_1)$ (h was first introduced by Poincaré, and its definition and construction is given in Mather's paper [6].) But first we observe that we may assume that g is defined on the infinite cylinder $S^1 \times \mathbb{R}$: outside of the annulus A we simply define it as

$$g(x, y) = \begin{cases} g(x, 0) + (y, 0), & \text{when } y < 0 \\ g(x, 1) + (y, 0), & \text{when } y > 1. \end{cases}$$

With this definition any circle $S^1 \times \{y\}$ with $y \ge 1$ or $y \le 0$ is invariant under g.

Choose a lift $G : \mathbb{R}^2 \to \mathbb{R}^2$ of the map g, and let $h \in C^1(\mathbb{R}^2)$ be the generating function for G. Thus a sequence $\{x_k : k \in \mathbb{Z}\}$ is the sequence of x - coordinates of an orbit $\{(x_k, y_k) : k \in \mathbb{Z}\}$ of G if and only if it satisfies

$$\Delta(x_{k-1}, x_k, x_{k+1}) = 0 \quad (\forall k \in \mathbb{Z}) \tag{1}$$

where $\Delta(a, b, c) = h_2(a, b) + h_1(b, c)$, and the h_i denote the partial derivatives of h. The twist property of the map g guarantees that the function $\Delta(a, b, c)$ is strictly increasing in a and c, and the way we defined G outside of the annulus A is such that $\Delta(a, b, c) \to \pm\infty$ if $c \to \pm\infty$ or $a \to \pm\infty$ (this follows from the fact that G has the "infinite twist property," i.e. that $\lim_{y\to\pm\infty} pr_1(G(x, y)) = \pm\infty$.)

In [1] we studied recurrence relations like (1), and our main observation was that they may be considered as a discrete analog of a second order elliptic PDE, meaning that one can use Perron's method of sub- and super harmonic functions to construct solutions of (1).

Thus we showed that if one defines *a subsolution* for (1) to be a sequence $\underline{x}_k$ with $\Delta(\underline{x}_{k-1}, \underline{x}_k, \underline{x}_{k+1}) \ge 0$ for all $k \in \mathbb{Z}$, and likewise a super solution to be sequence with $\Delta(\overline{x}_{k-1}, \overline{x}_k, \overline{x}_{k+1}) \le 0$ $(\forall k \in \mathbb{Z})$, then, *given any sub solution* $\underline{x}_k$ *and super solution* $\overline{x}_k$ *which satisfy* $\underline{x}_k \le \overline{x}_k$ $(\forall k \in \mathbb{Z})$ *there exists a solution* x_k *of* (1) *with* $\underline{x}_k \le x_k \le \overline{x}_k$ $(\forall k \in \mathbb{Z})$. The proof of this statement is entirely elementary.

Using this general method for constructing solutions from sub- and super solutions, we proved the following criterion for positivity of the topological entropy of the map g:

Proposition. *If there exist a sub solution* $\underline{x}_k$ *and a super solution* $\overline{x}_k$ *for* (1) *with*

$$\limsup_{k\to\infty} \frac{\underline{x}_k}{k} \le \omega_0 \qquad \liminf_{k\to-\infty} \frac{\underline{x}_k}{k} \ge \omega_1 \tag{2'}$$

and

$$\limsup_{k\to-\infty} \frac{\overline{x}_k}{k} \le \omega_0 \qquad \liminf_{k\to\infty} \frac{\overline{x}_k}{k} \ge \omega_1 \tag{2''}$$

for some $\omega_0 < \omega_1$, then $h_{top}(g) > 0$. In fact, there is a compact $K \subset A$ which is invariant under some iterate g^q of g, such that $g^q|K$ has a Bernoulli shift as a factor.

The proof is again elementary, and exploits the fact that the recurrence relation (1) is invariant under the $\mathbb{Z}^2$ action $(\tau_{m,\,n}x)_k = x_{k-m} + n$ on the space of biinfinite sequences $\mathbb{R}^{\mathbb{Z}}$. Using this invariance one can construct a large number of sub and super solutions by taking suprema and infima of translates $\tau_{m,\,n}(\underline{x})$ and $\tau_{m,\,n}(\overline{x})$ of the given sub and super solution. The Perron - process then provides one with a lot of solutions of (1) for which the corresponding orbits of the map g constitute the set K of the proposition.

The idea behind our proof of theorem A is the following. Assume the existence of an orbit $\{(x_n, y_n) \in S^1 \times [0,1] : n \in \mathbb{Z}\}$ of g which "connects the lower boundary A_0 of the annulus with its upper boundary A_1," in the sense that $\lim_{n\to\infty} y_n = 1$ and $\lim_{n\to-\infty} y_n = 0$. Corresponding to this orbit we have the sequence of x - coordinates $\{\underline{x}_n : n \in \mathbb{Z}\}$ of the lifted orbit $\{(\underline{x}_n, y_n) \in \mathbb{R}\times[0,1] : n \in \mathbb{Z}\}$ where $\underline{x}_n \mod \mathbb{Z} = x_n$. If we choose $\omega_0 < \omega_1$ so that the rotation number of $g|A_0$ is less than ω_0, and the rotation number of $g|A_1$ is more than ω_1, then the sequence $\{\overline{x}_n : n \in \mathbb{Z}\}$ is a solution of (1) which satisfies the inequalities (2"). If one also assumes that there is an orbit $\{(x_n', y_n') : n \in \mathbb{Z}\}$ going the other way (i.e. $\lim_{n\to-\infty} y_n' = 1$ and $\lim_{n\to\infty} y_n' = 0$), then its corresponding sequence of x - coordinates $\{\underline{x}_n : n \in \mathbb{Z}\}$ is a solution of (1) which satisfies the inequalities (2'). By the proposition the map g then must have positive topological entropy.

Unfortunately, it's not clear to me whether any of the two orbits $\{(\underline{x}_n, y_n) : n \in \mathbb{Z}\}$, $\{(\overline{x}_n, y_n) : n \in \mathbb{Z}\}$, should exist. However, a theorem of Birkhoff's provides us with finite orbit segments which have approximately the same behaviour as the orbits whose existence we just presumed. Below we'll show that one can extend the x_n - sequences corresponding to Birkhoff's finite orbit segments, in such a way that they become sub- and super solutions satisfying (2), so that we can apply the proposition to complete the proof of theorem A. The way we extend the x_n - sequences follows an idea of Hall and Boyland's.

Let $[\sigma_0,\, \sigma_1]$ be the rotation interval of the map g on the annulus A, and choose *rational* numbers $\omega_0, \omega_1 \in (\sigma_0,\, \sigma_1)$ with $\omega_0 < \omega_1$. Without loss of generality we may assume that $\sigma_0 > 0$, and hence that all rotation numbers involved are positive. It follows from the work of J. N. Mather and S. Aubry that there exist Birkhoff orbits with rotation numbers ω_0 and ω_1 respectively. If we denote the corresponding sequences of x - coordinates by $\underline{z}_k$ and $\overline{z}_k$ ($k \in \mathbb{Z}$), then the fact that these sequences come from Birkhoff orbits implies that they are monotone, i.e.

$$\underline{z}_k < \underline{z}_{k+1} \qquad \overline{z}_k < \overline{z}_{k+1},$$

and that they are periodic in the sense that

$$\underline{z}_{k+q_0} \equiv \underline{z}_k + p_0, \qquad \overline{z}_{k+q_1} \equiv \overline{z}_k + p_1$$

for all $k \in \mathbb{Z}$ (where $\omega_j = p_j/q_j$ and $\gcd(p_j, q_j) = 1$).

Consider any orbit $\{(x_n, 0) : n \in \mathbb{Z}\}$ of the map G restricted to the lower boundary of the annulus. Since $G|A_0$ is a circle homeomorphism, the sequence x_n must be monotone: $x_n < x_{n+1}$ ($\forall n \in \mathbb{Z}$). Then there exists an integer n_0 (independent of the orbit under consideration), such that for some $k \in \{0, \ldots, n_0 - 1\}$ and some $l \in \mathbb{Z}$, one has

$$\underline{z}_l \le x_k < x_{k+1} \le \underline{z}_{l+1}.$$

Indeed, if this were not true, then any interval $(\underline{z}_l, \underline{z}_{l+1})$ would contain at most one x_i, which contradicts the fact that the rotation number σ_0 of $G|A_0$ is less than the rotation number ω_0 of the Birkhoff orbit corresponding to the $\underline{z}_k$'s.

By continuity there is a $\delta > 0$ such that the same holds for any orbit segment of the map G with length n_0, which stays in the (narrow) strip $S_0(\delta) = \mathbb{R} \times (0, \delta)$.

After increasing n_0, if necessary, the same arguments show that we may assume that for any orbit segment $\{(x_i, y_i) : 0 \le i \le n_0\}$ of the map G which is contained in the strip $S_1(\delta) = \mathbb{R} \times (1-\delta, 1)$ there exist $0 \le k < n_0$ and $l \in \mathbb{Z}$ with

$$x_k \le \overline{z}_l < \overline{z}_{l+1} \le x_{k+1}.$$

Birkhoff showed that in a zone of instability for an area preserving twist map there exist orbits which stay arbitrarily long near one of the boundary components, then wander around in the annulus (in some unspecified way), and finally stay arbitrarily long near the other boundary component. More percisely, he showed that there is an orbit segment $\{(x_n, y_n) : -N \le n \le N\}$ for the map G, such that $(x_n, y_n) \in S_0(\delta)$ for $-N \le n < -N + n_0$, and $(x_n, y_n) \in S_1(\delta)$ for $N - n_0 < n \le N$.

We had chosen the neighbourhoods $S_0(\delta)$ and $S_1(\delta)$ so small and the number n_0 so large that there exist $n_- \in \{-N, \dots, -N + n_0 - 1\}$ and $n_+ \in \{N - n_0, \dots, N-1\}$ and also $k_\pm \in \mathbb{Z}$ such that

$$\underline{z}_{k_-} \le x_{n_-} < x_{n_-+1} \le \underline{z}_{k_-+1} \tag{3}$$

and

$$x_{n_+} \le \overline{z}_{k_+} < \overline{z}_{k_++1} \le x_{n_++1}. \tag{4}$$

Now we can define our super solution:

$$\overline{x}_j = \begin{cases} \underline{z}_{k_-+j-n_-} & \text{if } j \le n_-, \\ x_j & \text{if } n_- < j \le n_+, \\ \overline{z}_{k_+j-n_+} & \text{when } j > n_+. \end{cases}$$

For all $j \notin \{n_-, n_-+1, n_+, n_++1\}$ the sequence $\overline{x}_j$ certainly satisfies $\Delta(\overline{x}_{j-1}, \overline{x}_j, \overline{x}_{j+1}) = 0$, since $\overline{x}_j$ and $\overline{x}_{j\pm1}$ coincide with the x - coordinates of an orbit of G. For $j = n_-$ we have $\overline{x}_{j-1} = \underline{z}_{k_--1}$, $\overline{x}_j = \underline{z}_{k_-}$ and $\overline{x}_{j+1} = x_{n_-+1} \le \underline{z}_{k_-+1}$, by (3). So, since $\Delta(a, b, c)$ is a monotone increasing function of c, we find that

$$\Delta(\overline{x}_{j-1}, \overline{x}_j, \overline{x}_{j+1}) \le \Delta(\underline{z}_{k_--1}, \underline{z}_{k_-}, \underline{z}_{k_-+1}) = 0.$$

Similar arguments show that for the remaining three values of j ($n_- + 1$, n_+ and $n_+ + 1$) one also has $\Delta(\overline{x}_{j-1}, \overline{x}_j, \overline{x}_{j+1}) \le 0$, so that $\overline{x}_k$ is indeed a super solution for (1).

For large values of j the sequence coincides with either $\underline{z}_k$ or $\overline{z}_k$, so that it is clear that $\overline{x}_j$ satisfies the inequalities ($2''$).

We could repeat the whole procedure to produce a sub solution $\underline{x}_j$ of (1) which satisfies the inequalities ($2'$): together the sub and super solution $\underline{x}_j$ and $\overline{x}_j$ allow us to apply the proposition and conclude that theorem A is true.

References

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