Chapter 11
What Content Knowledge Should We Expect in Mathematics Education?

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Abstract Two content topics will be described. One deals with geometric measurement: length, area, and volume. These are important topics and they have not been learned very well. In particular, what types of relations can one consider when dealing with these three aspects of geometric measurement? The second topic is fractions, and different views of what students should be expected to learn. A third topic will be briefly discussed. This is a new book which has many mathematical topics which arose in secondary classes, with comments on the content both as it relates to student learning and sometimes how the mathematics fits into what had previously been studied in earlier grades and also how the ideas fit into later material. The aim of the work was to set up a framework on mathematical content knowledge for teaching mathematics in secondary school. The authors asked for comments from readers, so some will be given.

11.1 Introduction

Roger Howe has not only done very important work in mathematics, but he has taken on the task of trying to do important work in mathematics education at two ends, early elementary school and college level. This is rare and even rarer to succeed at both levels. His elementary school level work includes the following (Howe, 2012): *Three Pillars of First Grade Mathematics*. This illustrates very important aspects of whole numbers and addition and subtraction. Here is a comment following this article:

> *I have been an elementary math coach for 10 years and an elementary teacher for 29 years before that. We are always working on these concepts, but I have never read such an excellent, explicit article that pushes all the important understandings for the primary students.*

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At the college level, Barker and Howe wrote Barker & Howe (2007), which can be used as a text for a college course in geometry. Parts of this book can influence what teachers should know in detail about geometry, and some other parts contain material secondary level mathematics teachers should be familiar with.

11.2 Concepts and Skills

One of the recommendations from the National Mathematics Advisory Panel (2008, p. xix) is:

> *To prepare students for Algebra, the curriculum must simultaneously develop conceptual understanding, computational fluency, and problem solving skills. Debates regarding the relative importance of these aspects of mathematical knowledge are misguided. These capabilities are mutually supportive, each facilitating learning of the others.*

Here is another recommendation (National Mathematics Advisory Panel, 2008, p. xxi):

> *The mathematics preparation of elementary and middle school teachers must be strengthened as one means for improving teachers' effectiveness in the classroom. This includes preservice teacher education, early career support, and professional development programs. A critical component of this recommendation is that teachers be given ample opportunities to learn mathematics for teaching. That is, teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching and the connections of that content to other important mathematics, both prior to and beyond the level they are assigned to teach.*

Let me add a little to the last. In addition to mathematical knowledge for teachers, there are others who need more mathematical knowledge. Mathematics support people, both those in schools and in many other places, need deeper knowledge and a more comprehensive view of the curriculum, how it developed in earlier grade bands, and where it is going in later years. Mathematics educators have an important role to play in the preservice education of teachers, in their research, and in knowing more about the topics Lee Shulman mentioned teachers need to know (Shulman, 1986). One of these is knowledge of curriculum. With respect to this topic, let me just mention one example which is common to many textbooks on algebra and geometry.

Definition *Two nonvertical lines are parallel if and only if their slopes are equal. Two nonvertical lines are perpendicular if and only if the product of their slopes is* -1.

All of the groups mentioned above should know the difference between a theorem and a definition. Since parallel and perpendicular have geometric definitions, this should be a theorem, not a definition. Some people from the groups listed above should have written publishers and requested an appropriate change. As someone who has done this in person at displays at meetings and in letters to publishers, this seems to be something which surprises people. Some staff agree that a change should be made, but some defend what was written. Finally, with the Common Core (CCSSO and NGA, G-GPE.5), this is now listed as a theorem, as it should have been all along. The lead author of the geometry book containing the above *Definition* has a Ph.D. in mathematics from a major US university and clearly should know the difference between a definition and a theorem, so the problem is not just content knowledge, but expectations and taste. I am not mentioning which book this is since many books have this error.

This paper will point out some more examples where both conceptual understanding and computational skills are far too weak.

11.3 Geometric Measurements

In the book *Accessible Mathematics* (Leinwand, 2009, p. 92), Steven Leinwand wrote the following:

> *For this reason, effective instruction balances a focus on conceptual understanding (such as the meaning of area and perimeter and how they are related) with a focus on procedural skill (such as how to find the area and perimeter of plane figures).*

This is a book written for elementary and middle school teachers. Let us first address the issue of area and perimeter, a common topic in primary school. A problem devised by Deborah Ball and used later by Liping Ma needs to be mentioned about teacher knowledge concerning a possible connection between area and perimeter. The story is told that a student comes to class very excited. She has figured out a new property that the teacher has never told the class. She said that she has discovered that as the perimeter of a rectangle increases, the area also increases, and illustrates this with two examples, a square of side 4 cm and a rectangle with sides 4 cm and 8 cm. The question for teachers is: How do you respond?

If you have not read Ma's description of the results when this question was asked of some teachers in the USA and in China, let me strongly suggest you read Chap. 4 in Ma (2010). There were 23 US teachers, 12 of them had 1-year experience of teaching, and the other 11 averaged 11 years of teaching. Two of these teachers accepted the student's claim, three investigated the claim, and one was able to show that the claim was false. The rest said they would look it up, often because they did not remember formulas for the perimeter and/or the area of a rectangle. Ma interviewed 72 Chinese elementary school teachers. About the same percent accepted the claim. The rest worked on the problem, and about 70% of them were able to show that the student's claim was false. One gave a very cogent answer (Ma, 2010, p. 97):

> *The area of a rectangle is determined by two things, its perimeter and its shape. The problem of the student was that she only saw the first one. Theoretically, with the same perimeter, let's say 20 cm, we can have infinite numbers of rectangles as long as the sum of their lengths and widths is 10 cm. For example, we can have* $5 + 5 = 10$, $3 + 7 = 10$, $0.5 + 9.5 = 10$ *even* $0.01 + 9.99 = 10$, etc., etc. *Each pair of addends can be the two sides of a rectangle. As we can imagine, the area of these rectangles will fall into a big range. The square with*

sides of 5 cm will have the biggest area, 25 square cm, while the one with a length of 9.99 and a width of 0.01 will have almost no area. Because in all the pairs of numbers with the same sum, the closer the two numbers are, the bigger the product they will produce.

A number of years ago, I read a manuscript of a book which had the following problem:

Find rectangles whose area is equal to its perimeter.

To show that this problem made no sense, I suggested considering a rectangle with sides 3 and 6. The numbers one gets for perimeter and area are both 18. Consider the unit length to be 1 foot to begin with, and then change to yards. The dimensions of the rectangle are now 1 yard and 2 yards so that the perimeter is 6 and the area is 2. Then change to inches and one gets $2(36 + 72) = 216$ for the perimeter and $36 \times 72 = 2592$ for the area. So, are the perimeter and area the same or is one larger? One now sees that none of these is true; one cannot compare lengths with areas since their units of measurement are different. I omitted feet and square feet which suggests that these cannot be compared. However, one can compare the square of length with area, and for rectangles this is a problem which can be given and solved in late primary school or middle school. It is closely related to the claim made by the Chinese teacher of the square having the largest area among all rectangles with the same perimeter. The usual way this isoperimetric (same perimeter) problem is used in the USA is when the area of a rectangle has only been discussed when the side lengths are positive integers. Then, for many specific examples, there are only finitely many cases to consider and these can be treated by computation.

There are a number of different ways to treat the general case. One is to take a rectangle with sides $a > b$ and use it to construct much, but not all, of the square with side $(a + b)/2$ by removing a smaller rectangle with sides $(a - b)/2$ and b and placing it on top of the remaining rectangle. A small square of side $(a - b)/2$ is missing, so the area has decreased by this much. What has been done here is to give a geometric proof of the identity:

$$((a+b)/2)^2 - ab = ((a-b)/2)^2.$$

Since the right-hand side is positive when $a \neq b$, we have shown that $((a + b)/2)^2 \geq ab$. Divide by $(2(a + b))^2$ the square of the perimeter for the original rectangle and the square to get

$$ab/(2(a+b))^2 \leq 1/16.$$

This shows that the area A of the original rectangle divided by the square of the perimeter P satisfies

$$A/P^2 \leq 1/16 \tag{11.1}$$

and equality holds only when the rectangle is a square.

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This is a direct relationship between area and perimeter for a rectangle but clearly not what was being asked for. I think it is too much to expect many teachers to go this far, but the description Ma quoted from a Chinese teacher is something we could hope for. We are very far from this now. I would like math specialists and high school math teachers to know all of this. In a new book (Wu, 2016, p. 197), Wu includes the isoperimetric inequality (11.1) for rectangles as a problem. This is the first of two books he has written for middle school teachers.

Teachers in elementary and middle schools need to know some things about lengths and areas of figures that are not necessarily rectilinear. There is a difference between lengths and areas in a plane, which is important but almost completely ignored in school mathematics. When dealing with area, one can find, intuitively, both upper and lower approximations to the area, and, for standard figures showing up in school mathematics, these can be refined to get good approximations to the area of the figures. For arc length in the plane, if one connects successive points on a curve, one gets a lower approximation to the length of the curve, and for smooth curves, this is what will be used later in calculus to motivate the formula for the arc length of a curve. However, an upper bound on the arc length, even for a circle, requires work and is not appropriate for school mathematics.

How much of this do you think Leinwand had in mind, and how much do you think most readers of his book would know? The following will give some idea. Near the start of Chap. 9 in Leinwand (2009), there is the following:

> *What is the formula for the volume of a sphere? Really, do you know it? Have you forgotten it? Do you ever use it? Do you even care? ... But now return to our middle school and high school classes where memorizing and regurgitating the formula* $V = (4/3)\pi r^3$ *is a perfect way to sort students out on the basis of memorization criteria that have little relation to understanding and actually using the formula.*

Here is what he seems to think *understanding* this formula is:

> *...knowing how much 4/3 is, what* π *is equal to, what the r represents, and what that little elevated 3 means – that is how to use the formula once it is presented.*

He could at least have mentioned why the exponent is 3, since volumes of similar figures change as the cube of the factor of dilation. He could also have mentioned the fact that the constant π is the same constant which appears in the formulas for the area and circumference of a circle. Mathematics has many miracles, some minor and some major, and these should be celebrated. The area of a circle is a constant times r^2 and the circumference is a constant times r. The fact that these constants differ only by a factor of 2 is a miracle which can be motivated relatively easily. The fact that the same constants in the volume of a sphere and its surface area are also rational numbers times the same constant for a circle is a bit deeper. It is possible to define a four-dimensional sphere, and here the relevant constant is a rational multiple of π^2. I do not expect most mathematics educators to know this last part, but the two- and three-dimensional results are miracles and should be appreciated by people doing mathematics education. The factor of 1/3 is also interesting; it comes from a cone which, in school mathematics, can be motivated from a pyramid, but that is too far of a stretch to mention in a book like the one Leinwand wrote. However, it should be much more relevant in a high school class than to suggest that one needs to ask students “how much 4/3 is.”

Let us return to the problem of perimeter and area of rectangles. Here is another way to treat this problem at an older age. It is often useful to revisit a problem after new ideas have been introduced. The following is an example of this. If the sides of a rectangle have lengths a and b, then the area is ab and the perimeter is $2(a + b)$. Let us simplify the last to $a + b$ and form a function as follows:

$$f(x) = x^2 + (a + b)\,x + ab = (x + a)\,(x + b)\,.$$

Setting this equal to 0 gives a quadratic equation with real roots. We know a necessary and sufficient condition for this, which for this equation gives

$$(a + b)^2 - 4ab \geq 0 \text{ and equal to 0 if and only if } a = b.$$

Recall that the inequality we just proved is the one obtained before when considering the isoperimetric inequality for rectangles. One aspect of this argument is an example of an important fact, the connections between the coefficients in a quadratic polynomial and the zeros of this polynomial. This connection also holds for polynomials of higher degrees. This suggests the question: Is there is a similar theorem in three dimensions? It is much more likely that an algebraic method would generalize easily than that a cutting and pasting argument like the one in two dimensions would. The argument below contains aspects which all high school teachers would benefit from, and some middle school teachers would also.

The natural analogue of a rectangle is a rectangular prism, or a box to give it a shorter name. If the edge lengths are a, b, c, then the volume is abc, the surface area is $2(ab + ac + bc)$, and the sum of the edge lengths is $4(a + b + c)$. Again, we will form a function using the simplified products; that is, the coefficients 2 and 4 will be dropped. Set

$$g(x) = (x + a)\,(x + b)\,(x + c) = x^3 + (a + b + c)\,x^2 + (ab + ac + bc)x + abc.$$

The zeros of this function are real, and the last time we had that we used the criteria for a quadratic equation to have real zeros. We can do the same thing by reducing the cubic to a quadratic by a method which gives real roots to the quadratic. Fortunately there is a result like this in calculus. A real-valued differentiable function which has two zeros has a zero of the derivative which lies between the two zeros of the function. This is Rolle’s theorem. Next,

$$g'(x) = 3x^2 + 2\,(a + b + c)\,x + (ab + ac + bc)\,.$$

This function has real zeros if and only if $4(a + b + c)^2 - 4 \times 3(ab + ac + bc) \geq 0$. As before, this inequality can be rewritten as

$$2\,(ab + ac + bc)\,/16(a + b + c)^2 \leq 1/24. \tag{11.2}$$

The left-hand side is the surface area of the box divided by the square of the sum of the edge lengths, and the constant on the other side is the same ratio for all cubes, so in particular for the cube which has the same sum of the edge lengths. Also, there is equality only if the given box is a cube. Thus for a box with fixed sum of the edge lengths, the largest surface area this box could have is when it is a cube. There should two more theorems of this type. If the sum of the edge lengths is given, the largest volume a box can have is when it is a cube, and if the surface area of the box is given, the largest the volume can be is when the box is a cube.

We would like to use a similar proof, but to do that, we need an operator which reduces the degree of a polynomial by one and does not remove the coefficient *abc*. Here are two ways this can be done. One is to reverse the coefficients in $g(x)$ by setting $h(x) = x^3 g(1/x)$. The other way is to make $g(x)$ homogeneous by introducing a new variable y:

$$h(x, y) = x^3 + (a + b + c)\, x^2 y + (ab + ac + bc)\, xy^2 + abcy^3.$$

Then take a derivative with respect to y. I will leave this problem now, so that the reader can have some fun with it. The basic idea used here is implicit in the work of Newton (1972) and explicit in a paper by Maclaurin (1729). Some of the connections with isoperimetric inequalities were stated by Hardy, Littlewood, and Polya (1952, p. 36), as are readable treatments of the inequalities of Newton and Maclaurin. The full story works in n-dimensions. The three-dimensional theorems are interesting for another reason. All that is needed to prove them is Rolle's theorem and a little algebra. When Rolle's theorem is done in calculus, it is just used as a step to getting the mean value theorem, which is then used in various ways. It is nice to have an application of Rolle's theorem which students can appreciate for its own sake. A paper (Askey et al., 2015) on the three-variable inequalities has appeared in Mathematics Teacher.

11.4 Fractions

Leinwand's book deals mostly with suggestions for teaching. His *Instructional Shift 8* is:

> *Minimize what is no longer important, and teach what is important when it is appropriate to do so.*

Here is what he wrote about fractions (Leinwand, 2009, p. 56):

> *Sevenths and ninths. When was the last time you encountered a seventh or a ninth in everyday life? Because nearly all encounters with fractions are limited to ruler fractions such as 1/2, 1/4, 1/8, and 1/16, thirds and sixths, and fifths and tenths, one has to question the need to find a common denominator for fifths and elevenths. Only in a textbook in a math class do we impose the lunacy of 3/13 + 4/7!*

For someone like me who read the original NCTM Standards back in the early 1990s, this might bring back memories of a longer paragraph which includes not only *small denominators* but the following (National Council of Teachers of Mathematics, 1989, p. 96):

This is not to suggest, however, that valuable instruction time should be devoted to exercises like 17/24 + 5/18 or 5 3/4 × 4 1/4, which are much harder to visualize and unlikely to occur in real-life situations.

As bad as the NCTM paragraph is, what Leinwand wrote is worse. What conceptual understanding could one have of fractions if one cannot find a common denominator for fifths and elevenths? Is it really significantly harder to add *3/13 + 4/7* than to add *3/4 + 4/5*? One is *(3 × 7 + 13 × 4)/(13 × 7)* and the other is *(3 × 5 + 4 × 4)/(4 × 5)*. The first is then *(21 + 52)/91 = 73/91* and the second is *(15 + 16)/20 = 31/20*. All of the numerical computations can be done mentally. If one wants to go one step further and write both as an integer and a fraction between 0 and 1, the second has an extra step, which adds a bit to the complexity so the two computations have about the same complexity. At least the NCTM example of addition had denominators which had some common factors, so finding the least common divisor adds to the complexity. Fortunately, the Common Core does not suggest that least common denominators be used when starting to add fractions with different denominators. One reason for not mentioning least common denominators when starting to add fractions with different denominators is that doing that means having to introduce two new ideas at the same time. It is usually much harder to learn two new things at the same time than to learn them separately.

One referee of this paper suggested dropping the section on fractions. Let me add a bit to help explain why I think it is necessary to focus on this topic. First, here are some results on an eighth grade TIMSS fraction problem from 2011.

Which shows a correct method for finding 1/3–1/4?

A. (1 – 1)/(4 – 3)
B. 1/(4 – 3)
C. (3 – 4)/(3 × 4)
D. (4 – 3)/(3 × 4)

Here are a few of the results on this question. The numbers are percents:

	Correct	A	B	C	D
Average	37.1	25.4	26.0	9.4	37.1
Korea	86.0	2.7	6.9	4.2	86.0
USA	29.1	32.5	26.1	10.7	29.1
Finland	16.1	42.3	29.5	8.7	16.1

You can read more results at http://tinyurl.com/z118a7u or (Askey, 2015) and links provided there.

I hope most readers are as concerned about this as I am. After the first few NAEP results, there were articles written about how poorly our students did on fractions. One is on the web and it contains other references. See Post (1981). One item was 1/2 + 1/3 and 33% of the students got it right. Unfortunately the alternate possible responses were not included, and I have been unable to find them on the NAEP website. There are other results mentioned in this paper, and the highest score was 74% of 8th grade students correctly picking the answer to 4/12 + 3/12.

There are some other fraction problems given in TIMSS which set up a quandary both with the question about 1/3–1/4 and among the results there. Here are two questions and a few results. These are from TIMSS 1995 and were given to students in grades 7 and 8.

K9 $3/4 + 8/3 + 11/8 =$

A 22/15 B 43/24 C 91/24 D 115/24

The eighth grade international average was 49% correct. The international average was 35% for answer A. For the USA it was 42% for A and 45% for D. Notice that both A and B have answers which are smaller than 2, and the sum is clearly larger than 2 since the second fraction is 2 2/3. 11/8 = 1 3/8 > 1 3/9 = 1 1/3, so the sum of the second and third fractions is larger than 4, but 91/24 < 4 since 24*4 = 96. Those picking A clearly did not realize this type of argument, but it is likely that some of the others did since the percent of students picking B or C was small.

Here is another problem which involves subtraction.

L17 What is the value of 2/3–1/4–1/12?

A 1/6 B 1/3 C 3/8 D 5/12 E 1/2

For both the USA and internationally, the most common answer was the correct one, B: 39% USA and 42% internationally. The second most popular response was D, with 25% USA and 26% internationally. My guess as to why this was the second most popular answer is the denominator. Contrast this problem with the 1/3–1/4 one. It is much harder to do the calculations in an exam setting, and there were five possible answers rather than 4 so one might expect a larger percent of students to be able to answer the 1/3–1/4 problem correctly. Yet 39% of the students did the more complicated calculation correctly, while only 29% gave the correct answer for 1/3–1/4. When I wrote the comments on the 2011 TIMSS question which was linked earlier, I knew that two of the wrong answers were picked because their forms were somewhat like what one would do for a whole number subtraction problem. I wrote that the other incorrect answer might have some reason for picking it, but did not mention what I now think is a reason, and this reason also helps explain why all three wrong answers might be picked. There is a relatively new book (Kahneman, 2011) by Daniel Kahneman, *Thinking, Fast and Slow*. Kahneman argues that humans have two levels of thinking, which he calls System 1 and System 2. The first is what is used initially, and if it comes up with an answer which seems reasonable, that is

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usually as far as the thinking goes. If not, the second system is used. The first might come up with any of the four answers. The two with answers equivalent to 0 and 1 can be explained by analogy with whole numbers. The other two could occur in the following ways. Suppose a student knows that there is a formula for how to find the difference between two fractions and it is a somewhat messy formula. These two answers are somewhat messy so a student who is unsure what to do might pick one of these two answers at random. If 10% did this and got a negative number, I would expect that about 10% would have gotten the correct answer with no more sure knowledge, so the 29% correct could be replaced by about 19% who knew enough to do the calculation correctly via a rapid system in their brain. Here I am assuming that the problem of 1/3–1/4 seems so simple that few students will have to think hard about how to come up with an answer that requires the slower part of the brain. That would not be the case for 2/3–1/4–1/12. There is a way to do this rapidly, $1/4 + 1/12 = 3/12 + 1/12 = 4/12 = 1/3$, and $2/3–1/3 = 1/3$, but few of our students will do this. For the 1995 problems, the popular incorrect answer for the addition problem clearly comes in a way similar to what Kahneman describes, and I suspect that the most common mistake for the subtraction problem came in a similar way for some students, but most will have had to do some thinking about how the subtraction is actually done. Of course, just some thinking might not be enough for students to do the calculation correctly.

11.5 Mathematical Understanding for Secondary Teaching

There is a recent book with the title of this section (Heid et al., 2015). This is part of a long-term project of mathematics educators at the Pennsylvania State University and the University of Georgia. Their goal was to start to map out for secondary mathematics what has been done for primary school mathematics under the name of Mathematical Understanding for Secondary Teaching, or as they summarize, MUST. This book (Heid et al., 2015) contains 43 situations which arose in school classes and comments on the mathematics involved or, in some cases, that could have been involved in the lesson or as important background information for teachers and supervisors. These examples and general knowledge were used to develop the MUST framework. There are three general categories: Mathematical Proficiency, Mathematical Activity, and Mathematical Context of Teaching. The first has five parts carried over from *Adding It Up* (National Research Council, 2001), Conceptual Understanding, Procedural Fluency, Strategic Competence, Adaptive Reasoning, and Productive Disposition, and a sixth has been added, Historical and Cultural Knowledge. The first chapter, written by Jeremy Kilpatrick, ends with:

> *Just as we have sought the input of many mathematicians, mathematics teachers, and teacher educators during construction of this framework, we welcome comments on our final product from those in the field.*

There are many comments which could be given, but here only one situation will be discussed. This is Chap. 41, Situation 35, *Calculation of Sine*.

The prompt is: *After completing a discussion on special right triangles (30°-60°-90° and 45°-45°-90°), the teacher showed students how to calculate the sine of various angles using a calculator. A student then asked*: *"How could I calculate sin(32°) if I do not have a calculator."*

Various methods are described: the definition of sine for a right triangle with the use of a protractor and ruler or with dynamical geometry software, a secant line using 30° and 45°, a tangent line from 30°, and a Taylor polynomial of degree 7 of *sin(x)* about 0. There is a very important problem where this question arose and was solved in a different way by Ptolemy. Greek trigonometry dealt with chords in circles rather than triangles because this was the setting for uses in astronomy. Tables needed to be computed, and two things were available: chords associated with angles of 36° as well as the angles mentioned previously and how to find the lengths of a chord of half the length of a given chord in terms of the length of the given chord. Ptolemy's theorem could be stated and a little information given about how he used it to get good approximations for that early time. In the course of outlining this, it would be clear that Ptolemy's theorem was used as we would now use the easier addition formulas for sine and cosine. Teachers and some students would then see two important aspects of the history of mathematics. Very good work was done long ago, and when learning about it, one can often see how problems led to development of new mathematical results. One can also frequently see how it is now easier to do what had been done because of later work. Twenty years ago I did not know Ptolemy's theorem nor how useful it could be as a source of different proofs which use material high school teachers should know well, but based on experience in a course on proofs at the post calculus level, few good college students knew this material. Here is an illustration of how to complicate what should be a very simple proof.

From (Wikipedia, 2017), the proof of Ptolemy's theorem is easily reduced to proving the following trigonometric identity:

$$\sin(a+b)\sin(b+c) = \sin(a)\sin(c) + \sin(b)\sin(a+b+c).$$

This is followed with: "Now by using the sum formulae, $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$, it is trivial to show that both sides are equal to" [and then a complicated expression which is the sum of four terms each of which is the product of four factors is given]. Look it up to see if you think this is trivial. However, if one uses a simple corollary of the cosine formula, $\cos(x-y) - \cos(x+y) = 2\sin(x)\sin(y)$, one easily sees that both sides are equal to $[\cos(a-c) - \cos(a+2b+c)]/2$. When teaching this proof of Ptolemy's theorem, it was started in class and given as a homework problem to complete. A simple proof of the addition formulas had been given, but nothing had been said about how the product formulas followed from the full addition formulas. All of the students had taken the full calculus sequence and this included moderately complicated integrals, so they will have had problems where the product formulas would have been natural tools to use. A few had been able to work out a proof directly using the addition formulas, but none of them did it as slickly as the writer of the Wikipedia article did, and I do not consider that argument as trivial. It is tedious and ugly. None of them used the product formula to give a very nice proof.

I was very pleased that a sense of history was added to the MUST framework but a bit disappointed that no one thought of or knew enough to add Ptolemy's work as a good example of how history could be used to illustrate some important points.

11.6 Conclusions

I wish this could have been a more positive paper. The report (National Mathematics Advisory Panel, 2008) from the National Mathematics Panel Advisory group was very good, but it has basically been ignored almost since it was published. The final version of the Common Core Mathematics Standards (CCSS-M, 2010) is much better than I thought it would be, and parts of it could make a big difference if a few other things happen. One is adequate professional development. By and large that has not happened. Textbooks need to be improved. As Wu has remarked, it will take a lot of work to replace TSM [textbook school mathematics] by something much closer to mathematics. See (Wu 2017) for this and a much broader treatment of needed content knowledge than has been given here. The goal of the present paper is to illustrate to some extent the depth of the problems. The problems go well beyond content knowledge for teachers, it exists at all levels. Roger Howe has spent a lot of time and energy on whole numbers, and this has been appreciated. As school mathematics becomes more abstract, which it does with fractions, the problems become harder. Wu has spent a lot of time and energy on fractions, and the revised treatment of fractions in the Common Core is very similar to what he has been writing for about 15 years. However, the comments in Section 3 from Leinwand's book show that much more education needs to be done in the general mathematics education community. There is a review of Leinwand and Mathematics (2009) in the Spring 2015 issue of the *NCSM Newsletter*. The review includes the following:

> *The concluding chapters focus on accountability and provide a variety of practical expectations that rely on commitment from all levels to effect change. The book includes a lesson plan template, a crib sheet for raising mathematics achievement for all, and a research-based vision of teaching and learning with 12 interrelated characteristics of effective instruction in mathematics. This has become part of the introduction in my methods course for beginning middle level and high school mathematics teachers.*

There were no comments on the statements: *question the need to find a common denominator for fifths and elevenths* or *the lunacy of 3/13 + 4/7*. One wonders why such bad advice is almost never called out in the mathematics education literature. Leinwand's book (Howe) was published in 2009, and the first draft of these standards below high school was only released in late 2009, so Leinwand's book reflects his views before the Common Core Standards were available. One hopes it is somewhat different now.

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