ROTH’S THEOREM IN THE INTEGERS

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In the finite field setting we concerned ourselves to the linear equation \( x + y = 2z \), and again this is our primary concern, although it is of no extra difficulty to deal with more general forms \( a_1x_1 + \ldots + a_nx_n = 0 \) where the \( a_i \) are nonzero integers satisfying \( \sum_i a_i = 0 \) and \( n > 2 \). These types of equations are called translation invariant. From here on out we let \( l(x) = a_1x_1 + \ldots + a_nx_n = 0 \) be a fixed translation invariant equation in at least three variables. Solutions to \( l(x) = 0 \) are referred as trivial if \( x_1 = x_2 = \ldots = x_n \), in which case \( x \) is called a trivial point.

The upper density of a set \( A \subset \mathbb{Z} \) is given by

\[
\limsup_{N \to \infty} \frac{|A \cap [-N,N]|}{2N + 1}.
\]

Theorem A: (Roth’s Theorem). If \( A \subset \mathbb{Z} \) is a set of positive upper density then there is a nontrivial \( x \in A^n \) which solves the equation \( l(x) = 0 \).

The outline of the proof is the same as in the finite field case. We begin by proving result on the quantity

\[
T(\phi_1, \ldots, \phi_n) = \sum_{x \in [-N,N]^n} \phi_1(x_1) \ldots \phi_n(x_n) 1_H(x)
\]

where the \( \phi_i \) are functions on the integers bounded above by one in absolute value, and \( H = \{ x \in \mathbb{Z}^n : l(x) = 0 \} \).

Proposition 1. If \( n > 2 \) then we have

\[
|T_N(\phi_1, \ldots, \phi_n)| \lesssim ||\phi_1||_{L^\infty(\mathbb{R}/\mathbb{Z})} N^{n-2}
\]

where the implied constant may depend on the \( a_i \)’s and \( n \).

Proof. This result follows from a use of the Holder inequality, and as such we need to have a feel for the \( L^p(\mathbb{R}/\mathbb{Z}) \) norms of the \( \hat{\phi}_i \). This is achieved by interpolation. Recall that if we have \( 1 \leq s \leq t \leq \infty \) and

\[
\frac{1}{p} = \frac{\theta}{s} + \frac{1 - \theta}{t}
\]

for \( \theta \in [0,1] \) we have the \( L^p \) estimate

\[
||f||_{L^p} \leq ||f||_{L^s}^\theta ||f||_{L^t}^{1-\theta}.
\]

We have the estimates

\[
||\hat{\phi}_i||_{L^2(\mathbb{R}/\mathbb{Z})}^2 = ||\phi_i||_{L^2(\mathbb{Z})}^2 = \sum_{x \in [-N,N]} |\phi(x)|^2 \leq \sum_{x \in [-N,N]} 1 \approx N
\]
(by Plancherel), and
\[ ||\hat{\phi}_i||_{L^\infty(\mathbb{R}/\mathbb{Z})} \leq \sum_{x \in [-N,N]} |\phi(x)| \lesssim N \]

(straight from the definition of \( \hat{\phi}_i \) and the triangle inequality). Putting these things together gives the bound
\[ ||\hat{\phi}_i||_{L^p(\mathbb{R}/\mathbb{Z})} \lesssim N^{1-1/p} \]
if \( p \geq 2 \).

Now we may apply Holder’s inequality (with the decomposition 1 = \( \frac{1}{\infty} + \frac{1}{n-1} + \ldots + \frac{1}{n-1} \)) to \( T_N \) to obtain
\[
|T_N(\phi_1, \ldots, \phi_n)| \leq ||\hat{\phi}_1||_{L^\infty(\mathbb{R}/\mathbb{Z})} ||\hat{\phi}_2||_{L^{n-1}(\mathbb{R}/\mathbb{Z})} \ldots ||\hat{\phi}_n||_{L^{n-1}(\mathbb{R}/\mathbb{Z})} \lesssim ||\hat{\phi}_1||_{L^\infty(\mathbb{R}/\mathbb{Z})} (N^{1-1/(n-1)})^{n-1}
\]
which proves the proposition. \( \square \)

The main difference between the \( \mathbb{Z} \) and \( \mathbb{F}_p^d \) cases is that there are no subspaces of an interval. However arithmetic progressions are similar enough to intervals, and the next result shows that we can fragment our interval in progressions on long lent

**Lemma 1.** Let \( \xi \in \mathbb{T} \) and \( \epsilon > 0 \) be given. If \( N \) is large enough, depending on \( \epsilon \), then we have a partition
\[ [-N,N] \setminus E = \bigcup_i P_i. \]
where the \( P_i \) are disjoint arithmetic progressions of equal length, the set \( E \) is small, \( |E| \lesssim \epsilon^5 N \), and for each \( i \) and any \( x, y \in P_i \) we have
\[ |e(x\xi) - e(y\xi)| \leq \epsilon \] (0.1)

**Proof.** For fixed \( \xi \) we have coprime integers \( a \) and \( q \) with \( |\xi - a/q| \leq 1/qL \) for some \( q \leq L \). Set \( M = N^{1/4} \) and \( L = N^{1/3} \), and also work under the assumption that \( \epsilon > N^{-1/12} \). Then consider the progression
\[ P^* = \{0, q, 2q, \ldots, (M - 1)q\}. \]
We have
\[ |e(qx\xi) - 1| = |e(qx\beta) - 1| \leq \frac{M}{L} = N^{-1/12} < \epsilon. \]

The interval \( I = \{0, 1, 2, \ldots, qM - 1\} \) is covered by translates \( P^* + s \) \((s = 0, 1, \ldots, q - 1)\). We now try to cover \([-N,N] \) with translates of \( I \) of the form \( I + (-N) + jqM, (j \leq (2N+1)/qM) \) This can be done with the exception of at most \( qM \) elements of \([-N,N] \), and this collection is our set \( E \). As \( q \leq L \) we have \(|E| \leq \epsilon^5 \). Our set of progressions is then collection \( P^* + s + (-N) + jqM \) for \( s = 0, \ldots, q \) and \( j \leq (2N+1)/qM \).

Each progression outlined above is of the form \( P^* + z \) for some number \( z \). Then for \( x, y \in P^* + z, x > y \), we have some \( t \) and \( s \) with \( x = qt + z \) and \( y = qs + z \) for some \( t, s \in \{0, 1, \ldots, M\} \). Thus
\[ |e(x\xi) - e(y\xi)| = |e((qt + z)\xi) - e((qs + z)\xi)| = |e((q(t-s))\beta) - 1| \leq \epsilon \]
on each progression. \( \square \)
Lemma 2. Let $A \subset [-N,N]$ be a set with $\delta(2N+1)$ elements. If we have $||\hat{f}_A||_{L^\infty(\mathbb{R}/\mathbb{Z})} \geq \eta$, then there is an arithmetic progression $P$ of length $N^{1/4}$ such that

$$|A \cap P| \geq (\delta + \eta/4)|P|$$

provided $N \geq N_0(\eta)$.

Proof. This is similar to the finite field case. We have some $\xi \in \mathbb{R} \mathbb{Z}$ with

$$|\sum_{x \in [-N,N]} f_A(x)e(x\xi)| \geq \eta N.$$

Apply lemma 1 for this $\xi$ and some $\epsilon$ to be chosen later, and we have a collection of arithmetic progressions $\{P_j\}_{j=1}^J$ and a set $E$ which partition $[-N,N]$. On each $P_j$ we select a single point $x^{(j)}$, and this gives $|e(x\xi) - e(x^{(j)}\xi)| \leq \eta$ for any $x \in P_j$. Then we have

$$\sum_{x \in [-N,N]} f_A(x)e(x\xi) = \sum_{j=1}^J e(x^{(j)}\xi) \sum_{x \in P_j} f_A(x) + \sum_{j=1}^J \sum_{x \in P_j} f_A(x) (e(x\xi) - e(x^{(j)}\xi))$$

$$+ \sum_{x \in E} f_A(x)e(x\xi).$$

As

$$|\sum_{j=1}^J \sum_{x \in P_j} f_A(x) (e(x\xi) - e(x^{(j)}\xi))| \leq \sum_{j=1}^J \sum_{x \in P_j} \epsilon$$

and

$$\sum_{x \in E} f_A(x)e(x\xi) \leq |E|$$

it follows that

$$|\sum_{j=1}^J e(x^{(j)}\xi) \sum_{x \in P_j} f_A(x)| \geq (\eta - 2\epsilon)N.$$

Also we have

$$\sum_{j=1}^J \sum_{x \in P_j} f_A(x) = - \sum_{x \in E} f_A(x)$$

and the right hand side is at most $\epsilon N$ in absolute value. Then we have

$$|\sum_{j=1}^J e(x^{(j)}\xi) \sum_{x \in P_j} f_A(x) + \sum_{x \in P_j} f_A(x) \geq (\eta - 3\epsilon)N.$$

Proceeding now as in the finite field case we arrive at some progression $P_j$ with

$$\sum_{x \in P_j} f_A(x) \geq ((\eta - 3\epsilon)/2)(N/J) \geq (\eta/4)|P_j|$$

by choosing $\epsilon = \eta/6$, and this implies the result. □
In this result we see that $N_0(\eta)$ can be taken to be $(\eta/6)^{-12} = c\eta^{-12}$.

For a given linear form $l(x)$ in at least three variables and given set $A \subset [-N, N]$ with $|A| = \delta(2N + 1)$ we have on of three possibilities:

(1) $A$ is $(c\delta^2)$-pseudorandom and we have $c'\delta^n N^{n-1} - |A|$ non-trivial solutions, and this is greater than zero if $N > c\delta^{-n/(n-2)}(> c\delta^{-24})$.

(2) $N \leq c\delta^{-24}$.

(3) There is an arithmetic progression $P$ with $|P| \geq N^{1/4}$ such that $|A \cap P| \geq (\delta + c\delta^2)$

The proof of Theorem A amounts to iterating the following procedure: If 1) holds then we are fine. Otherwise, assume 2) does not hold, so 3) does, and set $A' = A \cap P$. Then, by rescaling and translating, we map $P \to [-N', N']$, where $N' \geq N^{1/4}/2$, and let the image of $A'$ be $A''$. Solutions in $A''$ provide solutions in $A$ due to translation invariance.

Going back and forth between 1) and 3) can happen at most $c\delta^{-2}$ times, and as long as

$$(N/2)^{4-(c\delta^{-2})} \geq c\delta^{-24}$$

we are guaranteed that 2) is never an issue. In terms of $\delta$ have this to be the case if

$$\delta \geq \frac{c}{\sqrt{\log \log N}}$$

for small $\delta$. And again a little more efficient counting in the density increment can remove the square root.