Cauchy-Riemann equations

Let $f(z)$ be a complex-valued function of one complex variable. What does it mean for f to be continuous? differentiable?

1. Continuity

What does it mean for f to be continuous at some number $(a+bi)$? Intuitively we would like to say that as z approaches $(a+bi)$, the values of f approach the value $f(a+bi)$. I will not bore you with the ϵ, δ definition which makes rigorous sense of the word "approaches" in the above sentence.

We can think of f as a pair of functions of two variables if we write $f(a+bi) = u(a,b) + v(a,b)i$ where u and v are real-valued functions of two real variables. It is easy to see that f is continuous at $(a+bi)$ if and only if both u and v are continuous at (a,b).

The usual properties of continuous functions still hold. For example, sums and products of continuous functions are continuous. The ratios of continuous functions are continuous except for the points at which denominator is zero. The exponential function is clearly continuous. All polynomials are continuous.

Example 1. The function $f(z) = \bar{z}$ is continuous everywhere. Indeed, we have $f(a+bi) = a - bi$ so $u(a,b) = a$ and $v(a,b) = -b$, which are both continuous functions.

We can also talk about limits of complex valued functions. As with continuity, the existence of a $\lim_{z\to(a+bi)} f(z)$ is equivalent to the existence of $\lim_{(x,y)\to(a,b)} u(x,y)$ and $\lim_{(x,y)\to(a,b)} v(x,y)$.

2. Differentiability

What does it mean for $f(z)$ to be differentiable at $a+bi$? We will use the same definition as in the case of one real variable.

Definition 1. A complex-valued function f is called differentiable at $a+bi$ if there exists a limit

$$\lim_{z\to(a+bi)} \frac{f(z) - f(a+bi)}{z - (a+bi)}$$

We are tempted to say that this will be equivalent to the differentiability of u and v at (a,b), but it would be *incorrect!* It will turn out that while differentiability of f implies differentiability of u and v, in fact we can say more.

In order to informally explain this phenomenon, let us rewrite the definition of differentiability as follows. In the real-variable case f is differentiable at a with derivative $f'(a)$ if

$$f(x) = f(a) + f'(a)(x-a) + small(x)$$

with the "error" $small(x)$ going to zero faster than $(x-a)$ as x approaches a.

Similarly, in the complex case we have

$$f(z) = f(a+bi) + f'(a+bi)(z-(a+bi)) + small(z).$$

What is important here is that we have a complex multiplication in this formula. Suppose $f'(a+bi) = A+Bi$. If $z = x+iy$ and $f(x+yi) = u(x,y)+v(x,y)i$ as before, we get

$$u(x,y)+v(x,y)i = f(x+iy) = f(a+bi)+(A+Bi)((x-a)+(y-b)i)+small(x,y)$$

$$= u(a,b)+v(a,b)i+\Big((A(x-a)-B(y-b))+(A(y-b)+B(x-a))i\Big)+small(x,y)$$

$$= \Big(u(a,b)+A(x-a)-B(y-b)\Big)+\Big(v(a,b)+A(y-b)+B(x-a)\Big)\,i+small(x,y).$$

When we look at the real and imaginary parts separately, we get

$$u(x,y) = u(a,b) + A(x-a) - B(y-b) + small(x,y)$$

$$v(x,y) = v(a,b) + B(x-a) + A(y-b) + small(x,y).$$

When we compare it with the definition of differentiability of u and v we see that u and v are differentiable at (a,b) and

$$u_x(a,b) = A,\ u_y(a,b) = -B,\ v_x(a,b) = B,\ v_y(a,b) = A.$$

In other words, in order for $f(z)$ to be differentiable as a complex-valued function we need to have the following two identities.

$$\boxed{u_x = v_y,\ u_y = -v_x}$$

These are called *Cauchy-Riemann equations*. In the other direction, it can be shown that if u and v are differentiable and satisfy Cauchy-Riemann conditions, then the function $f(x+yi) = u(x,y)+v(x,y)i$ is a differentiable function of one complex variable.

Example 2. Consider the function $f(z) = z^2$. We have $f(x+yi) = x^2 + 2xy\,i + y^2i^2 = (x^2-y^2) + 2xy\,i$. This gives $u(x,y) = x^2-y^2$ and $v(x,y) = 2xy$. Let us check the Cauchy-Riemann conditions. We have $u_x = 2x = v_y$ and $u_y = -2y = -v_x$, so C-R equations are satisfied. In fact, if we look closely we see the derivative $A+Bi = u_x + v_x\,i = 2x+2y\,i = 2z$ as one would expect from $(z^2)'$.

Example 3. Let us check that the exponential function defined in the last lecture is differentiable as a function of complex variable. We have $e^{x+yi} = e^x\cos y + e^x \sin y\, i$, so $u(x,y) = e^x\cos y$ and $v(x,y) =$

$e^x \sin y$. We have $u_x = e^x \cos y = v_y$ and $u_y = -e^x \sin y = -v_x$. Moreover, the derivative $A + Bi$ is $u_x + v_x\, i = e^x \cos y + e^x \sin y\, i = e^{x+yi}$. So we have $(e^z)' = e^z$ as promised.

Problem Set C3

1. Check that the function $f(z) = \bar{z}$ is *not* differentiable.
2. The function $f(x + yi)$ is given by $f(x + yi) = (x^4 - 6xy^2 + y^4) + (4x^3y - 4xy^3)\, i$. Is this function differentiable? Can we find a more compact form for $f(z)$?
3. The function $f(x+yi)$ is given by $f(x+yi) = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\, i$. Is this function differentiable? Can we find a more compact form for $f(z)$?
4. Assume that $f(z) = u(x, y) + v(x, y)\, i$ is differentiable in some region. It can be shown (but is quite difficult) that u and v have partial derivatives to any order. Show that u and v are harmonic, i.e. $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$.
5. We know that $f(z) = u(x, y) + v(x, y)\, i$ is differentiable for all z. We also know that $u(x, y) = 5x - 3y + 4$. Find the function $v(x, y)$.

Answers to these problems are on the next page.

Answers.

1. $u(x,y) = x$, $v(x,y) = -y$, so $u_x = 1 \neq v_y = -1$.

2. Yes; $f(z) = z^4$.

3. Yes; $f(z) = \frac{1}{z}$.

4. $(u_x)_x = (C-R) = (v_y)_x = (v_x)_y = (C-R) = (-u_y)_y$, so $u_{xx} + u_{yy} = 0$, and similarly for v.

5. $v(x,y) = 3x + 5y + constant$