Cauchy's Theorem

Let $f(z)$ be a complex-valued function and let $\gamma(t) = (x(t), y(t)),\ t \in [c, d]$ be a parametric curve on the plane. In what follows we will assume that f is continuous and γ are differentiable. We will also use notation $f(x+yi) = u(x, y)+v(x, y)\, i$, so that u and v are the real and imaginary parts of f.

Definition 1. The integral

$$\int_\gamma f(z)dz$$

is a complex number given by

$$\int_\gamma \Big(u(x, y)dx - v(x, y)dy\Big) + \int_\gamma \Big(u(x, y)dy + v(x, y)dx\Big)\, i.$$

We remark on the motivation behind this definition. If we formally write $dz = dx + dy\, i$ and multiply $(u(x, y) + v(x, y)\, i)(dx + dy\, i)$, we get

$$\Big(u(x, y)dx - v(x, y)dy\Big) + \Big(u(x, y)dy + v(x, y)dx\Big)\, i$$

Another way of interpreting this is by plotting a number of points $z_0, \ldots, z_n$ on the curve γ and then taking the limit of

$$f(z_1)(z_1 - z_0) + f(z_2)(z_2 - z_1) + ... + f(z_n)(z_n - z_{n-1})$$

as n grows and z_i get closer to each other. After a fair amount of work one can show that the limit equals the expression in Definition 1, if f and γ are "nice enough".

It is important not to confuse this definition with that of the work of the vector field $\overrightarrow{(u, v)}$ along γ. Here the result is a complex number, and there it was a real number. Here we complex multiply $(u + v\, i)$ and $dx + dy\, i$ and there we dot multiply $\overrightarrow{(u, v)}$ and $\overrightarrow{(dx, dy)}$.

It is natural to wonder whether the Fundamental theorem of calculus works in the complex-valued case. The answer is *yes*. Let $F(z)$ be a differentiable complex-valued function and let $f(z) = F'(z)$. Let $\gamma(t),\ t \in [c, d]$ be a parametric curve.

Theorem 1.

$$\int_\gamma f(z)dz = F(\gamma(d)) - F(\gamma(c)).$$

Proof. If $F(x+yi) = U(x,y) + V(x,y)\,i$, then $f(x+yi) = U_x(x,y) + V_x(x,y)\,i$. We get

$$\int_\gamma f(z)dz = \int_\gamma \big(U_x dx - V_x dy\big) + \int_\gamma \big(U_x dy + V_x dx\big)\,i.$$

We use the C-R equations for U and V to rewrite this as

$$\int_\gamma f(z)dz = \int_\gamma \big(U_x dx + U_y dy\big) + \int_\gamma \big(V_y dy + V_x dx\big)\,i$$

$$= \int_\gamma \vec{\nabla}\,U\cdot d\mathbf{r} + \Big(\int_\gamma \vec{\nabla}\,V\cdot d\mathbf{r}\Big)\,i = \big(U(\gamma(d)) - U(\gamma(c))\big) + \big(V(\gamma(d)) - V(\gamma(c))\big)\,i$$

$$= \big(U(\gamma(d)) + V(\gamma(d)\,i)\big) - \big(U(\gamma(c)) + V(\gamma(c)\,i)\big) = F(\gamma(d)) - F(\gamma(c)).$$

In the above calculations we have used the formula for the line integral of the gradient of a function. □

Functions of one complex variable is often a topic of a semester-long course at upper undergraduate or first-year graduate level. We are just barely scratching the surface here, but we can establish the key result known as Cauchy's theorem.

Theorem 2. **(Cauchy's Theorem)** Let $f(z)$ be a complex-valued function which is differentiable in some domain D in the complex plane. Let γ be the boundary of D, travelled so that D stays on the left. Then

$$\int_\gamma f(z)dz = 0.$$

Proof. It is a rather straightforward application of Green's Theorem.

$$\int_\gamma f(z)dz = \int_\gamma \Big(u(x,y)dx - v(x,y)dy\Big) + \int_\gamma \Big(u(x,y)dy + v(x,y)dx\Big)\,i.$$

$$\int\!\!\int_D (-v_x - u_y)\,dxdy + \Big(\int\!\!\int_D (u_x - v_y)\,dxdy\Big)\,i.$$

We now use C-R relations for u and v (which are valid since f is differentiable in D) to see that we are integrating zero functions. This gives $\int_\gamma f(z)dz = 0 + 0\,i = 0$. □

Example 1. The function $f(z) = z$ is differentiable inside the unit circle D in the complex plane. Let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ be the boundary of D travelled once counterclockwise. In this simple example we can show that $\int_\gamma f(z)\,dz = 0$ in three different ways.

- It is easy to see that f is differentiable for all z (in fact, $f'(z) = 1$). By Cauchy's Theorem, since f is differentiable in D and γ is the boundary of D travelled so that D is on the left, we get $\int_\gamma f(z)\,dz = 0$.
- It is easy to show that $F'(z) = z$ where $F(z) = \frac{1}{2}z^2$. Then by Theorem 1 we get

$$\int_\gamma f(z)\,dz = F(\gamma(2\pi)) - F(\gamma(0)) = F(1) - F(1) = 0.$$

- We can also check that $\int_\gamma f(z)\,dz = 0$ by the direct calculation from the definition. We have $u(x,y) = x$, $v(x,y) = y$, $x = \cos t$, $y = \sin t$, $dx = -\sin t\,dt$, $dy = \cos t\,dt$.

$$\int_\gamma f(z)\,dz = \int_\gamma \Big(u(x,y)dx - v(x,y)dy\Big) + \int_\gamma \Big(u(x,y)dy + v(x,y)dx\Big)\,i$$

$$= \int_{t=0}^{t=2\pi} \Big(xdx - ydy\Big) + \int_{t=0}^{t=2\pi} \Big(xdy + ydx\Big)\,i$$

$$= \int_{t=0}^{t=2\pi} \Big(\cos t(-\sin t) - \sin t \cos t\Big)\,dt + \int_{t=0}^{t=2\pi} \Big(\cos t \cos t + \sin t(-\sin t)\Big)\,dt\,i$$

$$= \int_0^{2\pi} (-2\sin t \cos t)\,dt + \int_0^{2\pi} (\cos^2 t - \sin^2 t)\,dt\,i$$

$$= \int_0^{2\pi} (-\sin 2t)\,dt + \int_0^{2\pi} \cos 2t\,dt\,i = \frac{1}{2}\cos 2t|_0^{2\pi} + \frac{1}{2}\sin 2t|_0^{2\pi}\,i = 0 + 0\,i = 0.$$

Problem Set C4

1. Calculate the integral of the function $f(z) = z^2$ over the curve $\gamma(t) = (t, t^2),\ t \in [0,1]$ directly from Definition 1.

2. Calculate the integral of the function $f(z) = z^2$ over the curve $\gamma(t) = (t, t^2),\ t \in [0,1]$ using Theorem 1.

3. Calculate the integral of the function $f(z) = \frac{1}{z}$ over the boundary of the unit circle $\gamma(t) = (\cos t, \sin t),\ t \in [0, 2\pi]$ using Definition 1.

4. Explain why the result of Problem 3 does not contradict Cauchy's Theorem.

5. Let $f(z)$ be a function of one complex variable which is differentiable for all z in the unit circle around $z = a + bi$ except for $z = a + bi$ itself. Show that for $\epsilon < 1$ the number

$$\frac{1}{2\pi i} \int_{\gamma_\epsilon} f(z)dz,$$

where γ_ϵ is the boundary of the circle of radius ϵ, is independent of ϵ. (This number is called *residue* of f at $a + bi$.)

Answers.

1. $-\frac{2}{3} + \frac{2}{3}\,i$

2. Same as in Problem 1.

3. $2\pi i$

4. Function f is not differentiable at all points inside the region, because it is not even defined at zero. Hence Cauchy's Theorem is not applicable.

5. We can apply Cauchy's Theorem to the region between the circles of radii ϵ and 1 around $a+bi$, since f is differentiable there. The inner boundary will be $-\gamma_\epsilon$ since it has to be travelled clockwise in order for the region to stay on the left. Hence $\int_{\gamma_1} f(z)\,dz - \int_{\gamma_\epsilon} f(z)\,dz = 0$. This shows that $\int_{\gamma_\epsilon} = \int_{\gamma_1}$ and is independent of ϵ.