

FINAL, MATH 341 - DECEMBER 18, 2007

NIGEL BOSTON

Answer eight of the ten questions. Each is worth the same number of points. Show your working. Full credit will not be given for the answer without any justification. Make sure you answer each part of each question. Do not write essays! Concise, “to-the-point” but complete answers are favored.

1. (a) Explaining your reasoning solve the system of linear equations given by $x - y + z = 3, 2x - z = 1, 3x + y - 3z = -1$.

$$\begin{pmatrix} 1 & -1 & 1 & 3 \\ 2 & 0 & -1 & 1 \\ 3 & 1 & -3 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 2 & -3 & -5 \\ 0 & 4 & -6 & -10 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -3/2 & -5/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mapsto$$

$$\begin{pmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & -3/2 & -5/2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ so } z = r \text{ (free variable), } x = (r + 1)/2, y = (3r - 5)/2.$$

Solution set is $\{((r + 1)/2, (3r - 5)/2, r) \mid r \in \mathbf{R}\}$.

(b) As a subset of \mathbf{R}^3 , what does the solution set look like?

A line.

(c) Without explicitly computing it, what would the solution set look like if the right hand side of the first equation changed to 3.00001?

The empty set.

2. (a) State the Subspace Theorem.

A subset W of a vector space V is a subspace if W is non-empty, closed under addition (i.e. $\mathbf{v}, \mathbf{w} \in W \implies \mathbf{v} + \mathbf{w} \in W$), and closed under scalar multiplication (i.e. $r \in \mathbf{R}, \mathbf{v} \in W \implies r\mathbf{v} \in W$).

(b) Let \mathbf{P}_n be the set of polynomials of degree at most n , where n is a non-negative integer. Show that \mathbf{P}_n is a subspace of $\mathbf{F}(\mathbf{R})$. What is its dimension?

First, note that \mathbf{P}_n is a subset of $\mathbf{F}(\mathbf{R})$. Applying (a), first note that $\mathbf{P}_n \neq \emptyset$ since $0 \in \mathbf{P}_n$. Second, note that if $\mathbf{v} = v_0 + v_1x + \dots + v_nx^n$ and $\mathbf{w} = w_0 + w_1x + \dots + w_nx^n$ are in \mathbf{P}_n , then $\mathbf{v} + \mathbf{w} = (v_0 + w_0) + (v_1 + w_1)x + \dots + (v_n + w_n)x^n$ and $r\mathbf{v} = (rv_0) + (rv_1)x + \dots + (rv_n)x^n$ are also in \mathbf{P}_n . Its dimension is $n + 1$ since $1, x, \dots, x^n$ form a basis.

(c) Show that $\mathbf{F}(\mathbf{R})$ has subspaces of every dimension $0, 1, 2, \dots$. If V is an infinite-dimensional subspace of $\mathbf{F}(\mathbf{R})$, must it equal $\mathbf{F}(\mathbf{R})$? Justify your answer.

By (b), it has subspaces of every dimension $2, 3, 4, \dots$. The subspace $\{\mathbf{0}\}$ has dimension 0 and the space of constant functions has dimension 1. The answer is no, since $\mathbf{F}(\mathbf{R})$ has many infinite-dimensional subspaces such as the subspace of continuous functions or differentiable functions or infinitely differentiable functions.

3. (a) Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a finite non-empty subset of a vector space W . Carefully define what it means (i) for S to span W and (ii) for S to be linearly independent.

(i) S spans W if every element of W can be written as a linear combination of elements of S , i.e. $r_1\mathbf{v}_1 + \dots + r_n\mathbf{v}_n$. (ii) S is linearly independent if $r_1\mathbf{v}_1 + \dots + r_n\mathbf{v}_n = \mathbf{0} \implies r_1 = 0, \dots, r_n = 0$.

(b) Suppose V is an m -dimensional vector space ($m \geq 2$) and that $T = \{\mathbf{w}_1, \dots, \mathbf{w}_{m-1}\}$ is a linearly independent subset of V . Can we adjoin a vector to T to get a basis for V ? If so, explain the main idea - if not, give a counterexample.

Yes. $\text{span } T \neq V$ since T is not a basis of V since V has dimension m . Pick a vector \mathbf{v} in V outside $\text{span } T$. Then $T \cup \{\mathbf{v}\}$ is linearly independent since otherwise $r\mathbf{v} + r_1\mathbf{w}_1 + \dots + r_{m-1}\mathbf{w}_{m-1} = \mathbf{0}$ either (if $r = 0$) contradicts linear independence of T or (dividing by $r \neq 0$) contradicts \mathbf{v} being outside $\text{span } T$. $T \cup \{\mathbf{v}\}$ has size m , so is a basis.

(c) Suppose $U = \{\mathbf{u}_1, \dots, \mathbf{u}_{m+1}\}$ is a subset of V that spans the m -dimensional vector space V . Can we delete a vector in U to get a basis for V ? If so, explain the main idea - if not, give a counterexample.

Yes. U is not linearly independent since U is not a basis of V since V has dimension m . Say $r_1\mathbf{u}_1 + \dots + r_{m+1}\mathbf{u}_{m+1} = \mathbf{0}$ holds with not all r_i zero. Say $r_1 \neq 0$. Then \mathbf{u}_1 is a linear combination of the other \mathbf{u}_i so $\mathbf{u}_2, \dots, \mathbf{u}_{m+1}$ spans V . It has size m so is a basis.

4. (a) State the Dimension Theorem (relating rank and nullity of a linear map). If T is a linear map from V to W , then $\text{rank } T + \text{nullity } T = \dim V$.

(b) Suppose $L : M(3, 5) \rightarrow \mathbf{R}^n$ is linear and onto. Determine $\text{rank } L$ and $\text{nullity } L$.

Since L is onto, $\text{rank } L = n$. By the Dimension Theorem, $\text{nullity } L = \dim M(3, 5) - \text{rank } L = 15 - n$.

(c) Define isomorphism. Is there a value of n for which we can conclude that L must be an isomorphism? Justify.

An isomorphism is a linear map that is both one-to-one and onto. L is already linear and onto. L is one-to-one if and only if $\text{nullity } L = 0$, which happens when $n = 15$. So the answer is yes, namely for $n = 15$.

5. Consider the function $T : \mathbf{P}_2 \rightarrow \mathbf{P}_2$ defined by $T(p(x)) = xp'(x)$.

(a) Show that T is linear.

$T(p + q) = x(p + q)'(x) = x(p'(x) + q'(x)) = xp'(x) + xq'(x) = T(p) + T(q)$; if $r \in \mathbf{R}$, then $T(rp) = x(rp)'(x) = xrp'(x) = rxp'(x) = rT(p)$.

(b) Show that $\{1, x - 1, (x - 1)^2\}$ is a basis for \mathbf{P}_2 .

Let $p(x) \in \mathbf{P}_2$. Then $p(x + 1) \in \mathbf{P}_2$, say $= a + bx + cx^2$. Replacing x by $x - 1$, $p(x) = a + b(x - 1) + c(x - 1)^2$ so they span and there are $\dim \mathbf{P}_2$ of them.

(c) Write down the matrix of T relative to this basis.

$T(1) = 0, T(x - 1) = x = (x - 1) + 1, T((x - 1)^2) = 2(x - 1)x = 2(x - 1)((x - 1) + 1) = 2(x - 1)^2 + 2(x - 1)$, so the matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

6. (a) Show that if \mathbf{v} is an eigenvector of the matrix A with associated eigenvalue λ , then it is also an eigenvector of A^n for any non-negative integer n . What is the associated eigenvalue?

We claim that $A^n \mathbf{v} = \lambda^n \mathbf{v}$ so that \mathbf{v} is an eigenvector with eigenvalue λ^n . The proof is by induction. True for $n = 1$ since that's given. If true for a given n , consider $A^{n+1} \mathbf{v} = A(\lambda^n \mathbf{v}) = \lambda^n A \mathbf{v} = \lambda^{n+1} \mathbf{v}$, taking care of the inductive step.

(b) Find the eigenvalues of $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$ and corresponding eigenvectors.

$\det(\lambda I - A) = (\lambda - 5)(\lambda + 4) - (-3)6 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$, so the eigenvalues of A are $2, -1$. For $\lambda = 2$, we want the nullspace of $\begin{pmatrix} -3 & -3 \\ 6 & 6 \end{pmatrix}$,

yielding $x + y = 0$, so e.g. $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. For $\lambda = -1$, we want the nullspace of $\begin{pmatrix} -6 & -3 \\ 6 & 3 \end{pmatrix}$, yielding $2x + y = 0$, so e.g. $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

(c) Let $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find $A^3 \mathbf{v}$.

We compute that $\mathbf{v} = 8\mathbf{v}_1 - 5\mathbf{v}_2$, so $A^3 \mathbf{v} = 8A^3 \mathbf{v}_1 - 5A^3 \mathbf{v}_2 = 64\mathbf{v}_1 + 5\mathbf{v}_2 = \begin{pmatrix} 69 \\ -74 \end{pmatrix}$.

7. (a) Define diagonal matrix. Define diagonalizable matrix.

A diagonal matrix is a square matrix such that $a_{ij} = 0$ if $i \neq j$. A diagonalizable matrix is a square matrix that is similar to a diagonal matrix.

(b) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Is A diagonalizable?

$\det(\lambda I - A) = \lambda^3 - 1$ so the only eigenvalue of A is $\lambda = 1$. $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$

$\begin{pmatrix} x - y \\ y - z \\ -x + z \end{pmatrix}$, so the eigenspace is given by $x = y = z$, so is only 1-dimensional. Thus

A does not have a basis of eigenvectors so isn't diagonalizable.

(c) Let $B = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Find P such that $P^{-1}AP = B$.

$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ works, since e.g. $P^{-1}AP\mathbf{e}_1 = P^{-1}A\mathbf{e}_1 = P^{-1}\mathbf{e}_3 = \mathbf{e}_2 = B\mathbf{e}_1$

and likewise for $\mathbf{e}_2, \mathbf{e}_3$

8. (a) Define orthonormal basis.

An orthonormal basis of an inner product space is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ such that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$ is 1 if $i = j$ and 0 otherwise.

(b) Show that if $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthonormal subset of the vector space V , then S is linearly independent.

Suppose $r_1\mathbf{v}_1 + \dots + r_n\mathbf{v}_n = \mathbf{0}$. Then for each i we compute $0 = \langle r_1\mathbf{v}_1 + \dots + r_n\mathbf{v}_n, \mathbf{v}_i \rangle = r_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + \dots + r_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle + \dots + r_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle = r_i$, so every r_i is 0.

(c) Find an orthonormal basis of \mathbf{P}_2 with respect to the inner product $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$.

Start with the basis $1, x, x^2$. $\langle 1, 1 \rangle = 1$ and so we set $\mathbf{v}_1 = 1$. $\langle 1, x \rangle = 1/2$ and so $\langle 1, x - 1/2 \rangle = 0$. $\langle x - 1/2, x - 1/2 \rangle = 1/12$ so set $\mathbf{v}_2 = \sqrt{12}(x - 1/2) = \sqrt{3}(2x - 1)$. $\langle 1, x^2 \rangle = 1/3$ and $\langle \sqrt{3}(2x - 1), x^2 \rangle = \sqrt{3}/6$ and we consider $x^2 - 1/3 - (\sqrt{3}/6)\sqrt{3}(2x - 1) = x^2 - x + 1/6$. Since $\langle x^2 - x + 1/6, x^2 - x + 1/6 \rangle = 1/180$, we set $\mathbf{v}_3 = \sqrt{180}(x^2 - x + 1/6) = \sqrt{5}(6x^2 - 6x + 1)$. The orthonormal basis is then $1, \sqrt{3}(2x - 1), \sqrt{5}(6x^2 - 6x + 1)$.

9. (a) Define determinant of an $n \times n$ matrix A . Give the formula for A^{-1} .

Defined inductively. For $n = 1$, $A = (a_{11})$ and $\det A = a_{11}$. In general, let A_{ij} be the $(n - 1) \times (n - 1)$ matrix obtained by deleting the i th row and j th column of A . Then $\det A = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}$. $A^{-1} = (1/\det A) \text{adj} A$, where the ij th entry of $\text{adj} A$ is the j th cofactor $(-1)^{i+j} \det A_{ji}$.

(b) State Cramer's rule.

The solution of $A\mathbf{x} = \mathbf{b}$ is given by $x_j = \det A_j / \det A$, where A_j is obtained by replacing the j th column of A by \mathbf{b} .

(c) Use Cramer's rule to find a solution of the system $3x - 4y = 2$, $2x - 3y = 1$.

$A = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$ has $\det = 3(-3) - (-4)2 = -1$. $A_1 = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$ has $\det = 2(-3) - (-4)1 = -2$. $A_2 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ has $\det = 3(1) - 2(2) = -1$. So $x = (-2)/(-1) = 2$ and $y = (-1)/(-1) = 1$.

10. State with reasoning whether each of the following statements is true or false.

(a) Every linear map from \mathbf{R}^2 to \mathbf{R}^3 is one-to-one.

False. For example the zero map.

(b) Every nontrivial vector space has infinitely many elements.

True. If it contains $\mathbf{v} \neq \mathbf{0}$, then it contains $r\mathbf{v}$ ($r \in \mathbf{R}$), which are all different.

(c) If A is an $m \times n$ matrix and B an $n \times m$ matrix, then AB and BA have the same determinant.

False. Suppose $A = \begin{pmatrix} a & b \end{pmatrix}$ and $B = \begin{pmatrix} c \\ d \end{pmatrix}$. Then $AB = (ac + bd)$ with determinant $ac + bd$, whereas $BA = \begin{pmatrix} ca & cb \\ da & db \end{pmatrix}$ with determinant $cadb - cbda = 0$.

Pick any a, b, c, d such that $ac + bd \neq 0$ gives you a counterexample.

(d) 0 is never an eigenvalue of a matrix.

False. Any non-invertible matrix A has eigenvalue 0 since $A\mathbf{x} = \mathbf{0}$ has a nonzero solution.

(e) A system of 2004 linear equations in 2007 unknowns never has solution set consisting of a line.

True. It either leads to an inconsistency (solution set empty) or there are at least 3 free variables.