MIDTERM, MATH 341 - OCTOBER 23, 2007

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YOUR NAME:

This is a 75-minute midterm exam. No notes, books, or calculators are permitted. Answer all five questions below (each is worth 20%). Show your working. Full credit will not be given for just the answer without any justification.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. (a) State the Subspace Theorem.

If S is a nonempty subset of a vector space V and S is closed under addition ($u, v \in S$ implies $u + v \in S$) and under scalar multiplication ($r \in \mathbf{R}, v \in S$ implies $rv \in S$), then S is a subspace of V.

(b) Suppose U and V are subspaces of the vector space W. Define $U + V$ to be $\{u + v \mid u \in U, v \in V\}$. Show by example that $U + V \neq U \cup V$.

Say $U = \{(x, 0\} \mid x \in \mathbf{R}\}, V = \{(0, y) \mid y \in \mathbf{R}\}$. Then $U + V = \{(x, y) \mid x, y \in \mathbf{R}\} = \mathbf{R}^2$ whereas $U \cup V$ is just the two axes.

(c) Is $U + V$ necessarily a subspace of W?

Yes. It's nonempty ($\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + V$), closed under addition ($u_1 + v_1, u_2 + v_2 \in U + V$ have sum $(u_1 + u_2) + (v_1 + v_2) \in U + V$), and under scalar multiplication ($r(u + v) = ru + rv \in U + V$), so by (a) $U + V$ is a subspace.

(d) Is $U \cup V$ necessarily a subspace of W?

Our example in (b) is not a subspace, since $(1, 0), (0, 1) \in U \cup V$ but their sum $(1, 1) \notin U \cup V$.

2. (a) Let M be the matrix $\begin{pmatrix} 1 & -2 & 3 & 1 \\ 3 & -4 & 5 & 3 \\ 2 & -3 & 4 & 2 \end{pmatrix}$. By elementary row operations convert M to reduced row-echelon form.

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ 3 & -4 & 5 & 3 \\ 2 & -3 & 4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -4 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Find the solution set for the linear system $x - 2y + 3z = 1,\ 3x - 4y + 5z = 3,\ 2x - 3y + 4z = 2$. As a subset of $\mathbf{R}^3$, what does this solution set look like?

By (a), $x - z = 1, y - 2z = 0$, so z is a free variable, say r, $x = 1 + r, y = 2r$. Solution set is $\{(1 + r, 2r, r) \mid r \in \mathbf{R}\} = \{(1, 0, 0) + r(1, 2, 1) \mid r \in \mathbf{R}\}$, i.e. the line through $(1, 0, 0)$ in the direction $(1, 2, 1)$.

(c) Find the solution set for the linear system $x - 2y + 3z = 1,\ 3x - 4y + 5z = 3,\ 2x - 3y + 4z = 1$.

The operations in (a) now lead to a bottom row $0, 0, 0, 1$, so the solution set is empty.

3. (a) Suppose we have 8 linear equations in 10 unknowns. Must this system have infinitely many solutions? Give a proof or a counterexample.

No. It can have no solutions, e.g. $x_1 = 0, x_1 = 1, ...$

(b) Can the solution set in (a) consist of a single line? Give an example or a proof that it cannot.

No. if you have solutions, you have at least two free variables in reduced row-echelon form.

(c) Find a polynomial that passes through the points $(1, 3), (2, 6), (3, 13)$. Is it unique?

Try $p(x) = ax^2 + bx + c$. Then $3 = a + b + c, 6 = 4a + 2b + c, 13 = 9a + 3b + c$, so $3 = 3a + b, 7 = 5a + b$, so $4 = 2a, a = 2$, so $b = 3 - 3a = -3$, so $c = 3 - a - b = 4$. $p(x) = 2x^2 - 3x + 4$ works. Not unique e.g. $p(x) + (x - 1)(x - 2)(x - 3)$ also works.

4. (a) Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in a vector space V such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans V. Show that $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ also spans V.

If $\mathbf{v} \in V$, then $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ $(a, b \in \mathbf{R})$, so $\mathbf{v} = \frac{a+b}{2}(\mathbf{v}_1 + \mathbf{v}_2) + \frac{a-b}{2}(\mathbf{v}_1 - \mathbf{v}_2)$.

(b) Suppose that S is a linearly independent subset of a vector space V and let T be a subset of S. Show that T is linearly independent.

Suppose $T = \{\mathbf{v}_1, ..., \mathbf{v}_m\}$ and $S = \{\mathbf{v}_1, ..., \mathbf{v}_{m+k}\}$. If $a_1\mathbf{v}_1 + ... + a_m\mathbf{v}_m = \mathbf{0}$, then $a_1\mathbf{v}_1 + ... + a_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + ... + 0\mathbf{v}_{m+k} = \mathbf{0}$. By linear independence of S, $a_1 = 0, ..., a_m = 0$, so we're done.

(c) Suppose that S is a linearly dependent subset of a vector space V and let T be a subset of S. Is T necessarily linearly dependent? Give a proof or a counterexample.

No. Suppose $S = \{1, 2\} \subseteq \mathbf{R}$ and $T = \{1\}$. S is linearly dependent but T is not.

5. (a) Find a basis of $M(2,2)$ that contains the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Pick a matrix not in their span, e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (all matrices in the span have 1st and 4th entries equal), and then a matrix not in the span of those three, e.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (all matrices in the span have 2nd and 3rd entries equal). These 4 matrices then form a basis, since they're linearly independent and $M(2,2)$ has dimension 4.

(b) Find the coordinates of $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ with respect to this basis.

This is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so the coordinates are $(1,-1,0,0)$.

(c) Is it true that a single vector can be added to any two vectors in $\mathbf{R}^3$ to get a basis of $\mathbf{R}^3$? Give a proof or a counterexample.

No. Suppose the two vectors are e.g. $(1,0,0)$ and $(2,0,0)$.