FINAL, MATH 441

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Answer all four questions below. Show your working. Full credit will not be given for just the answer without any justification. Even if you cannot prove one part of a question, you will still get credit if you use it to prove a later part of that question.

1. (i) What is $2^{101} \pmod{12}$? (ii) What is $99^{101} \pmod{100}$? (iii) What is 121212121 (mod 9)? (iv) What is 121212121 (mod 101)? [Hint: what is 100 (mod 101)?]

2. (i) Show that if $k = md$ where m is odd , then $2^k + 1$ is divisible by $2^d + 1$. [Hint: set $x = 2^d$.] (ii) Show that if $2^k + 1$ is prime, then k is a power of 2. We call $2^{2^n} + 1$ the nth Fermat number F_n. (iii) Show by mathematical induction that $F_n = F_0F_1F_2...F_{n-1} + 2$ for all $n \geq 1$. [Hint: show that $(F_n - 1)^2 = F_{n+1} - 1$.] (iv) Show that $(F_m, F_n) = 1$ if $m \neq n$. Why does this imply that there exist infinitely many primes?

3. (i) Find a subgroup H of order 4 of the group of units U_{17}. (ii) State Lagrange's Theorem. (iii) List all the cosets of H in U_{17}. (iv) Give a 1-1 homomorphism from H to the permutation group S_4.

4. (i) Find a greatest common divisor of x^4+2 and x^3+x+1 in $\mathbf{F}_3[x]$. (ii) Write this greatest common divisor as a linear combination of x^4+2 and x^3+x+1. (iii) We set $\alpha = [x]$ and so $\mathbf{F}_3[\alpha] = \mathbf{F}_3[x]/(x^3+x+1)$. How many elements does $\mathbf{F}_3[\alpha]$ have? (iv) Find at least 3 units and 3 zero divisors in $\mathbf{F}_3[\alpha]$ (if they exist).