1. (i) Powers of 2 (mod 12) go 2, 4, 8, 4, 8, 4, ... Since 101 is odd, we get 8. (ii) \(99^{101} \equiv (-1)^{101} \equiv -1 \pmod{100}\) (iii) \(121212121 \equiv 1+2+1+2+1+2+1+2+1 \equiv 4 \pmod{9}\). (iv) \(121212121 \equiv 21 + 21 \times 100 + 21 \times 100^2 + 21 \times 100^3 + 100^4 \equiv 21 - 21 + 21 - 21 + 1 \equiv 1 \pmod{101}\)

2. (i) If \(k = md\) and \(x = 2^d\), then \(2^k + 1 = 2^{md} + 1 = x^m + 1\), which is divisible by \(x + 1 = 2^d + 1\) since \(m\) is odd (use the root test). (ii) If \(k\) is not a power of 2, then it has an odd factor \(m\) and then \(2^k + 1\) is divisible by \(2^d + 1\), so \(2^k + 1\) is not prime. (iii) The hint holds since \((F_{n-1} - 1)^2 = (2^n)^2 = 2^{n+1}\). The claim is true for \(n = 1\) since \(F_1 = 5 = F_0 + 2\). Assume the claim is true for a particular \(n\), i.e. that \(F_0F_1...F_{n-1} + 2 = F_n\). Then \(F_0F_1...F_n = F_0F_1...F_{n-1}F_n = (F_n - 2)F_n = (F_n - 1)^2 - 1 = F_{n+1} - 2\) by the hint, implying that the claim is true with \(n\) replaced by \(n + 1\). This completes the induction. (iv) Suppose \(m < n\). If \(p\) is a common factor of \(F_m\) and \(F_n\), then by (iii) \(p\) divides \(F_n - F_0F_1...F_{n-1} = 2\). But any factor of \(F_m\) must be odd. So \(F_m\) and \(F_n\) have no common factor, other than 1. This implies there are infinitely many primes since \(F_n\) must be divisible by some prime that hasn’t appeared in the factorizations of \(F_0, F_1, ..., F_{n-1}\).

3. (i) We need an element of order 4 in \(U_{17}\). Looking at powers of 2, we get 2, 4, 8, -1, so 2 will have order 8 so \(2^2 = 4\) has order 4. Then \(H = \{[4], [-1], [-4], [1]\} = \{[4], [16], [13], [1]\}\). (ii) If \(H\) is a subgroup of the finite group \(G\), then the order of \(G\) divided by the order of \(H\) is an integer, namely the number of cosets of \(H\) in \(G\). (iii) For the cosets, \(H = \{[4], [16], [13], [1]\}\), \(2 * H = \{[[8], [15], [9], [2]\}, 3 * H = \{[12], [14], [5], [3]\}\), leaving as the remaining coset \{[6], [7], [10], [11]\}. (iv) By Cayley, we just look at the effect of multiplication by the elements of \(H\). Multiplication by \([1]\) maps \([1], [4], [-1], [-4]\) to \([1], [4], [-1], [-4]\). Multiplication by \([4]\) maps \([1], [4], [-1], [-4]\) to \([4], [-1], [-4], [1]\). Multiplication by \([1]\) maps it to \([-1], [-4], [1], [4]\). Multiplication by \([-4]\) maps it to \([-4], [1], [4], [-1]\). Relabeling, the homomorphism is \([1] \mapsto (1, 2, 3, 4), [4] \mapsto (2, 3, 4, 1), [-1] \mapsto (3, 4, 1, 2), [-4] \mapsto (4, 1, 2, 3)\).

4. (i) \(x^4 + 2 = x(x^3 + x + 1) + (2x^2 + 2x + 2)\), \(x^3 + x + 1 = (2x + 1)(2x^2 + 2x + 2) + (x + 2)\), \(2x^2 + 2x + 2 = (2x + 1)(x + 2) + 0\). So a greatest common divisor is \(x + 2\). (ii) \(x + 2 = (x^3 + x + 1) - (2x + 1)(2x^2 + 2x + 2) = (x^3 + x + 1) - (2x + 1)((x^4 + 2) - (x(x^3 + x + 1)) = (1 + x + 2x^2)(x^3 + x + 1) - (2x + 1)(x^4 + 2)\). (iii) \(F_3[\alpha]\) has 3^3 = 27 elements. (iv) \(x^3 + x + 1 = (x+2)(x^2 + x + 2)\), where \(x^2 + x + 2\) is irreducible (since it has no root). So \(\alpha + 2, \alpha^2 + \alpha + 2\) are zero divisors, as is say 2\(\alpha + 1\) (or any multiple of either of the factors of \(x^3 + x + 1\)). As for units, we need elements relatively prime to the factors of \(x^3 + x + 1\), so e.g. 1, 2, \(\alpha, \alpha^2, \ldots\).