1. Define a relation $\sim$ on the set of real numbers $\mathbb{R}$ by saying that $x \sim y$ if and only if $x - y$ is an integer. Is this relation an equivalence relation? If so, describe the equivalence class that contains 0.5 and describe the partition of $\mathbb{R}$ produced by the equivalence relation.

2. Show that every integer $x$ satisfies $x^3 \equiv x \pmod{3}$. Show by induction that $x^{3^n} \equiv x \pmod{3}$. What is the remainder on dividing $2^{28}$ by 3? (There is a quick way.)

3. Show that $\sqrt{3}$ is not a rational number. Let $\mathbb{Q}[\sqrt{3}]$ consist of all real numbers of the form $a + b\sqrt{3}$, where $a, b$ are any rational numbers. You may assume that $\mathbb{Q}[\sqrt{3}]$ is a ring. Assuming this, is it true that $\mathbb{Q}[\sqrt{3}]$ is a field?

4. Find integers $r, s$ such that $1 = 11r + 13s$. Find an integer $x$ satisfying $11x \equiv 3 \pmod{13}$. What are the units of $\mathbb{Z}/13\mathbb{Z}$?
(1) Need to check R,S,T.

R holds since \( x - x \) is an integer for every \( x \) in \( \mathbb{R} \). S holds since if \( x - y \) is an integer, say \( n \), then \( y - x \) is an integer, namely \(-n\). T holds since if \( x - y \) and \( y - z \) are integers, then \( x - z = (x - y) + (y - z) \) is also an integer.

\( x - 0.5 \) is an integer if \( x \) is an integer plus a half, so the equivalence class is \( \{..., -0.5, 0.5, 1.5, 2.5, ... \} \).

Likewise a typical subset in the partition is \( \{..., -1 + a, a, 1 + a, 2 + a, ... \} \) where \( a \) can be chosen in \([0, 1)\).

(2) If \( x \equiv 0, 1, 2 \pmod{3} \), then \( x^3 \equiv 0, 1, 2 \pmod{3} \) respectively, so \( x^3 \equiv x \pmod{3} \).

Let \( P(n) \) be the statement \( x^{3^n} \equiv x \pmod{3} \). We just showed \( P(1) \). Now assume \( P(n) \) is true. Look at \( x^{3^{n+1}} = (x^{3^n})^3 \). This \( \equiv x^3 \) by \( P(n) \). But by what we just showed, this \( \equiv x \pmod{3} \). So \( P(n + 1) \) is established. That completes the induction.

By what we just showed, \( 2^{27} \equiv 2 \pmod{3} \) and so \( 2^{28} \equiv 4 \equiv 1 \pmod{3} \), so the remainder is 1.

(3) Suppose \( \sqrt{3} = a/b \) where \( a, b \) are positive integers. Then \( 3 = a^2/b^2 \), so \( a^2 = 3b^2 \). The power of 3 dividing \( a^2 \) must be even, whereas the power of 3 dividing \( 3b^2 \) is odd. A contradiction, so \( \sqrt{3} \) is not rational.

Let \( a + b\sqrt{3} \) be a nonzero element. Then \( \frac{1}{a+b\sqrt{3}} = \frac{(a-b\sqrt{3})}{(a+b\sqrt{3})(a-b\sqrt{3})} = \frac{a-b\sqrt{3}}{a^2-3b^2} = \frac{a}{a^2-3b^2} + \frac{-b}{a^2-3b^2} \) so every nonzero element is a unit, so \( \mathbb{Q}[\sqrt{3}] \) is a field.

(4) \( 13 = 1 \cdot 11 + 2, 11 = 5 \cdot 2 + 1 \), so \( 1 = 11 - 5 \cdot 2 = 11 - 5 \cdot (13 - 11) = 6 \cdot 11 - 5 \cdot 13 \pmod{13} \), so take \( r = 6, s = -5 \). Taking \( \pmod{13} \), \( 1 \equiv 6 \cdot 11 \pmod{13} \). Multiplying by \( 3, 3 \equiv 18 \cdot 11 \pmod{13} \). So \( x = 18 \) works. Since 13 is prime, every nonzero element of \( \mathbb{Z}/13\mathbb{Z} \) is a unit.