

PRACTICE FINAL, MATH 441

NIGEL BOSTON

Answer all five questions below. Show your working. Full credit will not be given for just the answer without any justification.

1. (i) What is the greatest common divisor of 10 and 21? (ii) Find a solution to the equation $10x \equiv 1 \pmod{21}$. (iii) What is the order of 10 (mod 21)? (There are long and short ways of doing this.)

2. (i) Show that $2^m - 1$ divides $2^n - 1$ if m divides n . [Hint: set $x = 2^m$.] (ii) Deduce that if $2^n - 1$ is prime, then n is prime. (iii) Show that if n is prime and $2^n - 1$ is divisible by a prime p , then $p \equiv 1 \pmod{n}$. (iv) Show that $2^{11} - 1$ is not prime.

3. We know that $1/3 = 0.333\dots$, $1/7 = 0.142857142\dots$, $1/9 = 0.111\dots$, $1/11 = 0.090909\dots$ are repeating decimals with periods 1, 6, 1, 2 respectively. (i) Show that if p is any prime other than 2 and 5, and if the order of 10 (mod p) is d , then $1/p$ has decimal expansion with period d . (ii) Deduce that the period of the decimal expansion of $1/p$ divides $p - 1$. (iii) Is there a prime p for which this period is exactly 3? If so, which?

4. (i) What is the order of U_{20} , the group of units mod 20? (ii) What is the order of [3] in U_{20} ? (iii) Find all the cosets of the subgroup generated by [3]. (iv) How does this illustrate Lagrange's Theorem?

5. (i) Describe explicitly a field F with 8 elements. (ii) Suppose a is an element of F other than 0 and 1. Is a primitive? Why? (iii) Give the factorization of $x^8 - x$ in $F[x]$.