

MATH 580/780I FINAL, FALL 2006

NIGEL BOSTON

For full credit you must explain your reasoning. Each question is worth an equal amount.

1. (a) State the binomial theorem.
(b) Show that the product of any four consecutive positive integers is divisible by 24.
2. Use the Euclidean algorithm to compute the greatest common divisor d of 30 and 72. Write d as a linear combination of 30 and 72.
3. Find all solutions in positive integers of $16x + 20y = 200$.
4. Let $a = 2^5 3^3 5^2$ and $b = 2^4 3^5 7^3$. (a) Write down the prime factorization of the greatest common divisor of a and b . (b) Write down the prime factorization of the least common multiple of a and b . (c) What is the exact power of 3 dividing $a + b$?
5. (a) Show that $n! - 1$ is not a perfect square if $n > 2$.
(b) Find all primes p such that $4p + 1$ is a perfect square.
6. (a) What are the last two digits of $987654321^{123456789}$?
(b) Find digits X and Y such that $3XY4$ is divisible by 99. [Hint: $99 = 9 \times 11$.]
7. Find all integers that satisfy both $2x \equiv 5 \pmod{7}$ and $2x \equiv 2 \pmod{10}$.
8. (a) State Euler's theorem.
(b) Let $n \geq 2$. Find the order of $2 \pmod{2^n - 1}$. [Hint: find k such that $2^k \equiv 1 \pmod{2^n - 1}$ holds.] Why does it follow that n divides $\phi(2^n - 1)$?
9. Show that $a^{561} \equiv a \pmod{561}$ whatever a is. [Hint: $561 = 3 \times 11 \times 17$, $560 = 2^4 \times 5 \times 7$.]
10. (a) Define what it means for a function f to be multiplicative.
(b) Show that $\sigma(n) > n > \tau(n)$ if $n > 2$, where $\sigma(n)$ is the sum of the divisors of n and $\tau(n)$ is the number of divisors of n . [Hint: use the definitions of σ and τ .]