MATH 580/780I MIDTERM 1 SOLUTIONS, FALL 2006

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1. (a) Call this statement P_n. We prove it by induction.

First, P_1 is true since $1^2 = 1$ and $\binom{2(1)+1}{3} = 1$.

Second, suppose P_k is true. The LHS of P_{k+1} is

$1^2 + 3^2 + 5^2 + ... + (2k-1)^2 + (2k+1)^2 = \binom{2k+1}{3} + (2k+1)^2 = (2k+1)(2k)(2k-1)/6 + (2k+1)^2 = (2k+1)(2k(2k-1)+6(2k+1))/6 = (2k+1)(4k^2-2k+12k+6)/6 = (2k+1)(4k^2+10k+6)/6 = (2k+1)(2k+2)(2k+3)/6 = \binom{2k+3}{3}$. Done.

(b) State as in the book.

2. (a) $n^2 - 1 = (n-1)(n+1)$. Since n is odd, there are two cases. If $n = 4k+1$, then $n-1$ is divisible by 4 and $n+1$ divisible by 2; if $n = 4k+3$, then $n+1$ is divisible by 4 and $n-1$ divisible by 2. So $n^2 - 1$ is divisible by 2 times $4 = 8$.

(b) These primes are odd, so by part (a) we just need to show $n^2 - 1$ is divisible by 3 too, since $24 = 8 \times 3$. If $n = 3k+1$, then $n-1$ is divisible by 3; if $n = 3k+2$, then $n+1$ is divisible by 3. In either case, $(n-1)(n+1)$ is divisible by 3.

3. (a) State as in the book.

(b) Let $a = 8$ and $b = 4$. Then $a^2 = 64$ divides $b^3 = 64$ but 8 does not divide 4.

4. (a) $60 = 48 + 12$; $48 = 4(12) + 0$; $gcd(48, 60) = 12 = 60 - 48$. Take $x = -1, y = 1$.

(b) $lcm(48, 60) = (48)(60)/gcd(48, 60) = (48)(60)/12 = 4 \times 60 = 240$.

5. (a) Let there be x soccer teams and y rugby teams. $74 = 11x + 15y$.

(b) By trial and error, $x_0 = 4, y_0 = 2$ works.

(c) Since $gcd(11, 15) = 1$, the general solution is $x = 4 + 15t, y = 2 - 11t$. For x to be nonnegative, $t \geq 0$. For y to be nonnegative, $t \leq 0$. So $t = 0$ gives the unique feasible solution.

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