1. Let $A$ be a field.
   (a) Show that $A[[x]]$ is an integral domain.
   (b) Describe the ideals of $A[[x]]$. Is $A[[x]]$ a PID?
   (c) How many maximal ideals does $A[[x]]$ have? How many prime ideals does $A[[x]]$ have?
   (d) Does the nilradical of $A[[x]]$ equal its Jacobson radical?

2. Let $A$ be a ring of prime characteristic $p$.
   (a) If $f = a_0 + a_1 x + \ldots + a_n x^n + \ldots \in A[[x]]$, compute $f^p$.
   (b) Deduce that if $f$ is nilpotent, then so are $a_0, a_1, \ldots$.
   (c) Give a ring $A$ and $a_0, a_1, \ldots \in A$, each of which is nilpotent, but such that $f = a_0 + a_1 x + \ldots + a_n x^n + \ldots$ is not nilpotent. [Hint: use (a).]

3. Let $A$ be a ring. We endow the set Spec$(A)$ of prime ideals of $A$ with a topology by specifying the closed sets: for any ideal $I$ of $A$, let $V(I) = \{ P \in \text{Spec}(A) | I \subseteq P \}$.
   In this problem, verify that these subsets of Spec$(A)$ satisfy the axioms necessary to be the closed sets for a unique topology.
   (a) Show that $V(A) = \emptyset$.
   (b) Show that $V(\{0\}) = \text{Spec}(A)$.
   (c) Show that $V(I_1 \cup V(I_2) = V(I_1 \cap I_2)$.
   (d) Let $\{I_\alpha\}$ be a family of ideals. Let $I$ be the ideal they generate. Show that $V(I) = \cap V(I_\alpha)$.

4. (a) Show that $A \mapsto \text{Spec}(A)$ is a contravariant functor from the category of commutative rings with one to the category of topological spaces.
   (b) Let MaxSpec$(A) \subseteq \text{Spec}(A)$ be the subset consisting of all maximal ideals, given the subspace topology. Show that it need not be a closed subset.
   (c) Show, however, that MaxSpec$(A)$ is quasi-compact.

5. Let $S$ be an infinite set and $K$ a field. Then $A = K^S$ denotes the ring of all functions from $S$ to $K$.
   (a) For each $x \in S$, define $m_x$ to be the set of all functions $f \in A$ with $f(x) = 0$. Show that $m_x$ is a maximal ideal and that $A/m_x \cong K$.
   (b) By part (a), we have an embedding $S \to \text{MaxSpec}(A)$. Show that the induced topology on $S$ is discrete.
   (c) Why does it follow that there must exist other maximal ideals of $A$ besides the $m_x$? Can you find any?