1. Let $A$ be a ring and $S$ a subset closed under multiplication, containing 1. Recall that every ideal of $S^{-1}A$ is of the form $S^{-1}I$, where $I$ is an ideal of $A$.
   (a) For which $I$ is $S^{-1}I$ equal to $S^{-1}A$?
   (b) For which $I$ is $S^{-1}I$ a prime ideal of $S^{-1}A$? [Hint: look at the quotient ring.]
   (c) The natural ring homomorphism $A \rightarrow S^{-1}A$ induces a map from Spec$(S^{-1}A)$ to Spec$(A)$. Is it injective? What is its image?
   (d) Show that in particular Spec$(A_f)$ is homeomorphic to the basic open set $X_f$.

2. Let $A$ be a ring, $X = \text{Spec}(A)$. For each basic open $U$ we define a ring $A(U)$.
   (a) Set $A(X_f) = A_f$. Show that this is well-defined, i.e. if $X_f = X_g$, then $A_f = A_g$, so that $A(U)$ is unique up to a unique isomorphism.
   (b) Identify $A(\emptyset)$ and $A(X)$.
   (c) Suppose $U = X_f$ and $U' = X_g$ are basic open sets such that $U' \subseteq U$. How are $f$ and $g$ related? Use this to define a homomorphism $\rho : A(U) \rightarrow A(U')$. Show that $\rho$ depends only on $U$ and $U'$. Show that if $U = U'$, then $\rho$ is the identity map.
   (d) If $U'' \subseteq U' \subseteq U$, show that $\rho_{U',U''} \circ \rho_{U'',U'} = \rho_{U',U''}$.
   (e) Let $\varphi \in X$. Show that $\varprojlim A(U) \cong A_\varphi$, where the direct limit is over basic open sets $U$ containing $\varphi$.

3. (a) We showed that if $f : M \rightarrow N$ is an $A$-module homomorphism such that the induced map $f_\varphi : M_\varphi \rightarrow N_\varphi$ is an isomorphism for every prime ideal $\varphi$ of $A$, then $f$ is an isomorphism. Give an example of $A$-modules $M$, $N$ such that $M_\varphi \cong N_\varphi$ for every prime ideal $\varphi$, but $M$ and $N$ are not isomorphic.
   (b) Let $B = \mathbb{C}[x,y]/(xy)$ considered as an $A = \mathbb{C}[x]$-module. Show that for every maximal ideal $m = (x - a)$ of $A$ ($a \neq 0$), $B_m$ is free of rank 1 (so flat) over $A_m$, but for $a = 0$ it is not flat (because of torsion).

4. Which of the following rings are Noetherian? (Coefficients are in $\mathbb{C}$.)
   (a) The ring of rational functions of $z$ having no pole on the circle $|z| = 1$.
   (b) The ring of power series in $z$ with a positive radius of convergence.
   (c) The ring of power series in $z$ with an infinite radius of convergence.
   (d) The ring of polynomials in $z$ whose first 9 derivatives vanish at the origin.

5. Let $A$ be a Noetherian ring, $I$ an ideal, generated by $a_1,\ldots,a_r$. Say $x \in \cap I^n$.
   (a) Since $x \in I^n$, write $x = P_n(a_1,\ldots,a_r)$, where $P_n \in A[t_1,\ldots,t_r]$. Let $J_n$ be the ideal generated by $P_1,\ldots,P_n$. Show there exists $N$ such that $J_N = J_{N+1}$.
   (b) Let $P_{N+1} = Q_N P_1 + \ldots + Q_1 P_N$, show each $Q_i(a_1,\ldots,a_r) \in I$. Deduce $x \in I.$
   (c) Show that if $I$ is contained in the Jacobson radical of $A$, then $\cap I^n \neq \{0\}$. [Hint: idempotents.]