MATH 844: HOMEWORK 2, DUE FEB 9

2. (a) Prove that all the roots e_1, e_2, e_3 of $4x^3 - g_2x - g_3$ are real and distinct if and only if g_2 and g_3 are real and $\Delta := g_2^3 - 27g_3^2 > 0$.

(b) Suppose the conditions in (a) are met, and order the e_i so that $e_2 > e_3 > e_1$. Show that we can choose the periods of a lattice L such that $g_2 = g_2(L)$ and $g_3 = g_3(L)$, to be given by:

$$\omega_1/2 = i\int_{-\infty}^{e_1} 1/\sqrt{g_3 + g_2t - 4t^3}\ dt \qquad \omega_2/2 = \int_{e_2}^{\infty} 1/\sqrt{4t^3 - g_2t - g_3}\ dt$$

where we take the positive branch of the square root, and integrate along the real axis.

(c) Show that $\int_0^1 t^n/\sqrt{t(1-t)}\ dt = \frac{\pi.1.3.5...(2n-1)}{n!.2.2.2...2}$ for $n = 0, 1, 2, ...$

(d) Under the conditions of (a) and (b), set $\lambda = (e_3 - e_1)/(e_2 - e_1)$. Show that

$$\omega_2 = (1/\sqrt{e_2 - e_1})\int_0^1 1/\sqrt{t(1-t)(1-\lambda t)}\ dt$$

(e) Show that $\omega_2 = \pi(e_2 - e_1)^{-1/2}F(\lambda)$ where

$$F(\lambda) = \sum_{n=0}^{\infty}\left(\frac{1.3.5...(2n-1)}{2.2.2...2}\right)^2\lambda^n/(n!)^2$$