MATH 844: HOMEWORK 3, DUE FEB 16

3. Consider the problem of finding triples of positive integers (a, b, c) with no common factor such that the triangle ABC with sides a, b, c has the following property:

(∗) The median from A, the angle bisector from B, and the altitude from C meet in one point.

An example is $(12, 13, 15)$.

Setting $Y = 2b/(a+c), X = 2c/(a+c)$, show that $(*)$ gives $Y^2 = X^3 - 4X + 4$.

(a) Sketch the graph of the curve for $-3 \leq X \leq 3$. Is it an elliptic curve?

(b) Show that a point (X, Y) on this curve for which there exists a corresponding triangle ABC, must satisfy $0 < X < 2, 0 < Y < 2$.

(c) Call the curve E. Find $\{P \in E(\mathbf{Q}) \mid 6P = \infty\}$. [Hint: think geometrically.]

(d) There is a particularly simple triangle satisfying $(*)$, namely $(1, 1, 1)$. Show, however, that the corresponding point on E actually equals $-4Q$ where Q is the point $(2, 2)$.

(e) Show that $7Q$ also produces a solution to $(*)$.

(f) Assuming $\phi : E(\mathbf{R}) \cong \mathbf{R}/\mathbf{Z}$ as topological groups (to be shown in class), what property of $\phi(Q)$ is enough to ensure that multiples of Q produce infinitely many noncongruent triangles satisfying $(*)$?