Pro-$p$ groups and towers of rational homology spheres

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Abstract

In the preceding paper, Calegari and Dunfield exhibit a sequence of hyperbolic 3-manifolds which have increasing injectivity radius, and which, subject to some conjectures in number theory, are rational homology spheres. We prove unconditionally that these manifolds are rational homology spheres, and give a sufficient condition for a tower of hyperbolic 3-manifolds to have first Betti number 0 at each level. The methods involved are purely pro-$p$ group theoretical.

In [1], Calegari and Dunfield give a conditional answer to a question of Cooper [2, Problem 3.58] by exhibiting a series of hyperbolic 3-manifolds $M_1, M_2, \ldots$, such that

- The injectivity radius of $M_n$ is unbounded;
- Subject to the Generalized Riemann Hypothesis and Langlands-type conjectures about the existence of Galois representations attached to automorphic forms, $H^1(M_n, \mathbb{Q}) = 0$ for all $n$.

These 3-manifolds are constructed as quotients of hyperbolic 3-space by certain arithmetic lattices in $\text{SL}_2(\mathbb{C})$. In the following note, we explain how to prove unconditionally that $H^1(M_n, \mathbb{Q}) = 0$ for all $n$, without use of automorphic forms. The argument uses only the theory of pro-$p$ groups and should generalize to some other lattices in $\text{SL}_2(\mathbb{C})$. We emphasize, however, that the present argument applies is not in general a replacement for the argument of Calegari and Dunfield; we expect there will be many hyperbolic manifolds to which the method of Galois representations might be applicable, but whose fundamental groups do not have analytic pro-$p$ completion. In particular, it follows from results of Lubotzky [5, Thm 1.2, Rem 1.4] that when $\Gamma$ is a lattice with $\dim H^1(\Gamma, \mathbb{F}_p) \geq 4$, the pro-$p$ completion of $\Gamma$ is never analytic. On the other hand, the argument here does appear to apply to some non-arithmetic lattices, where it is difficult to see how the method of Galois representations could be pushed through.

Note: we use number theorists’ notation throughout, in which $\mathbb{Z}_3$ denotes the 3-adic integers, not the field with 3 elements.

We recall some basic facts about cocompact lattices in $\text{SL}_2(\mathbb{C})$. Let $\Gamma$ be a cocompact lattice. Then there is a number field $K$ (which can be taken to be the trace field of $\Gamma$) and a quaternion algebra $A$ admitting an injection $\Gamma \hookrightarrow A^\times$. (See [7, 3.2].) For each prime $p$ of $K$, let $A_p/K_p$ be the completion of $A$ at the prime $p$, and write $A^\times_p$ for the subgroup of elements of norm 1. If $U$ is a uniformly powerful subgroup of $A^\times_p$, the lower $p$-central series $U = U_0, U_1, U_2, \ldots$ is defined by $U_{i+1} = U_i^p [U, U_i]$. Write $H$ for hyperbolic 3-space; then $H/\Gamma$ is a compact hyperbolic 3-manifold, which is a rational homology sphere just when $H^1(\Gamma, \mathbb{Q}) = 0$.

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Proposition 1. Let $\Gamma$ be a cocompact lattice of $SL_2(\mathbb{C})$ and let $p$ be a prime of $K$ such that

- The norm of $p$ is an odd rational prime $p$;
- The closure of the image of $\pi : \Gamma \hookrightarrow A^\times_p$ contains a uniformly powerful open pro-$p$ subgroup $U$ of $A^\times_p$; we write $\Gamma_0$ for $\pi^{-1}(U)$.
- $\Gamma_0/\Gamma_0^p$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$.

Then every normal subgroup $H$ of $\Gamma_0$ with $p$-group quotient has $H^1(H, \mathbb{Q}) = 0$. In particular, taking $\Gamma_i$ to be $\pi^{-1}(U_i)$, the tower of compact $3$-manifolds $\mathcal{H}/\Gamma_i$ is nontrivial. It follows that the tower of compact $3$-manifolds $\mathcal{H}/\Gamma_i$ is nontrivial. It follows that the tower of compact $3$-manifolds $\mathcal{H}/\Gamma_i$ is nontrivial. It follows that the tower of compact $3$-manifolds $\mathcal{H}/\Gamma_i$ is nontrivial.

Proof. The unboundedness of the injectivity radii of $\mathcal{H}/\Gamma_i$ follows immediately from the fact that the $\Gamma_i$ have trivial intersection.

Write $T$ for the pro-$p$ completion of $\Gamma_0$. We will show that the surjection $T \to U$ is an isomorphism.

Since $U$ is a uniformly powerful open subgroup of a norm $1$ quaternion group over $\mathbb{Z}_p$, it has dimension $3$ (see [3, XII,a]). This implies in particular that $H^1(U, \mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^3$. It follows that the natural injection $H^1(U, \mathbb{Z}/p\mathbb{Z}) \to H^1(T, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism.

We will need the following easy criterion for injectivity of morphisms of pro-$p$ groups:

Proposition 2. Suppose $f : G_1 \to G_2$ is a surjective homomorphism of pro-$p$ groups such that

- $H^1f : H^1(G_2, \mathbb{F}_p) \to H^1(G_1, \mathbb{F}_p)$ is an isomorphism;
- $H^2f : H^2(G_2, \mathbb{F}_p) \to H^2(G_1, \mathbb{F}_p)$ is injective.

Then $f$ is an isomorphism.

Remark 3. All cohomology here is group cohomology with discrete coefficients.

Proof. Let $K$ be the kernel of $f$. Then the inflation-restriction exact sequence

$$H^1(G_2, \mathbb{F}_p) \hookrightarrow H^1(G_1, \mathbb{F}_p) \to H^1(K, \mathbb{F}_p)^{G_2} \to H^2(G_2, \mathbb{F}_p) \to H^2(G_1, \mathbb{F}_p)$$

shows that $H^1(K, \mathbb{F}_p)^{G_2}$ is trivial under the hypotheses of the proposition. But if $K$ is nontrivial, then $H^1(K, \mathbb{F}_p)^{G_2}$ is nontrivial — we may see this by considering a nontrivial finite quotient $K$ of $K$, observing that $H^1(K, \mathbb{F}_p)$ is a nontrivial $\mathbb{F}_p$-vector space, and recalling that the action of the pro-$p$ group $G_2$ on a nontrivial finite-dimensional $\mathbb{F}_p$-vector space necessarily has a nontrivial space of invariants. This proves the proposition. 

Now the cup product map

$$\wedge^2 H^1(U, \mathbb{F}_p) \to H^2(U, \mathbb{F}_p)$$

is surjective. This is a special case of a theorem of Lazard (see [8, Th 5.1.5]): if $G$ is a uniformly powerful pro-$p$ group of dimension $d$, with $p$ odd, the cohomology ring $H^*(G, \mathbb{F}_p)$ is a free exterior algebra generated by a basis $x_1, \ldots, x_d$ of $H^1(G, \mathbb{F}_p)$.

So let $a$ be an element in $H^2(U, \mathbb{F}_p)$ which vanishes in $H^2(T, \mathbb{F}_p)$. Then $a$ lifts to an element $\bar{a}$ in $\wedge^2 H^1(U, \mathbb{F}_p) = \wedge^2 H^1(T, \mathbb{F}_p)$ which vanishes under the cup product map to $H^1(T, \mathbb{F}_p)$. So in particular, if we can show that

$$\gamma : \wedge^2 H^1(T, \mathbb{F}_p) \to H^2(T, \mathbb{F}_p)$$
is injective, then $\tilde{a} = a = 0$, so the second condition of Proposition 2 is satisfied. But by [8, Theorem 5.1.6]), the injectivity of the cup product map is equivalent to the condition that $T/T^p$ is abelian, which is true by hypothesis.

Proposition 2 now implies that that $T \cong U$. By [3, III.9, XI a]), $U$ (and every open subgroup of $U$) is a $p$-group in the notation of [3, I.7]; so every open subgroup of $U$ has finite abelianization (and indeed has no proper infinite quotients whatever.) So the same is true for $T$; in particular, $\text{Hom}(K, \mathbb{Z}_p)$ is finite for each open $K \subset T$. But if $H$ is an open normal subgroup of $\Gamma_0$ with a $p$-group quotient, then the pro-$p$ completion $K$ of $H$ is an open normal subgroup of $T$, and $H^1(H, \mathbb{Z}_p) = H^1(K, \mathbb{Z}_p)$ is finite. This proves Proposition 1.

We now explain how to show that the tower of manifolds studied in [1] satisfies the conditions of Proposition 1. We recall some definitions and notation from [1]. Let $D$ be the quaternion algebra over $\mathbb{Q}(\sqrt{-2})$ which is ramified precisely at the two primes $\pi$ and $\bar{\pi}$ dividing 3, let $B$ be a maximal order of $D$, and let $B^\times$ be the group of units of $B$. Calegari and Dunfield consider a manifold $M_0$ whose fundamental group is isomorphic to $B^\times/\pm 1$.

Let $B_\pi$ be the maximal order in the completion of $D$ at $\pi$; then $B^\times_{\pi}$ is a profinite group with a finite-index pro-$3$ subgroup, and the natural map $B^\times \to B^\times_{\pi}$ is an inclusion whose image contains a dense subgroup of the group $B_1^\times_{\pi}$ of elements of reduced norm 1.

Let $Q$ be the unique maximal two-sided ideal of $B_\pi$; then $(1+Q^n) \cap B_1^\times$ is an open subgroup of $B_1^\times$ for all $n \geq 0$, and is a pro-$3$ group for $n \geq 1$. Let $\Gamma_n$ be the preimage of $(1+Q^n) \cap B_1^\times$ under $B^\times \to B^\times_{\pi}$. Then the content of [1, Theorem 1.4] is that $\Gamma_n$ has finite abelianization for all sufficiently large $n$.

In a slight discord of notation, the group denoted $\Gamma_2$ by Calegari and Dunfield plays the role of $\Gamma_0$ in Proposition 1. The closure of the image of $\Gamma_2$ is precisely the open subgroup $U = (1+Q^2) \cap B_2^\times$; this subgroup is powerful (as one can check by computing $U/U^3$) and torsion-free, and is thus uniformly powerful. So the second condition of Proposition 1 is satisfied, and it remains only to check that $\Gamma_2/\Gamma_3^2$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$.

One finds that the group $\Gamma_1$ has the presentation

$$\Gamma_1 := \text{Group < a, b, c, d | a*b^{-1}*c^{-1}*b*a^{-1}*d*c*d^{-1}, a*b*a^{-1}*d*c*d*a^{-1}*b*d^{-2}*c^{-1}*b^{-1}, c*d^2*c*d^2*c*d^2, c^3, a*c*b*c*a*b*d^{-2} >}$$

Then $\Gamma_2$ is the kernel of $\Gamma_1 \to \text{Hom}(\Gamma_1, \mathbb{Z}/3\mathbb{Z})$; one can easily compute a presentation of $\Gamma_2$ (too long to be worth including here) and from there it is a simple matter to compute $\Gamma_2/\Gamma_3^2$. We have thus shown that the manifolds appearing in [1] are all rational homology spheres.

**Remark 4.** The group $\Gamma_2$ does not have the congruence subgroup property [5]; however, one might think of Proposition 1 as asserting a kind of “pro-$3$ congruence subgroup property”: every finite-index subgroup of $\Gamma_2$ whose quotient is a 3-group is indeed congruence. It would be interesting to understand which lattices in $\text{SL}_2(\mathbb{C})$ are residual $p$-groups with the pro-$p$ congruence subgroup property for some $p$. This property certainly cannot hold for all lattices, since there exist lattices with infinite abelianization [4], [6].

**References**


