1. (a) Let $L/K$ be a finite Galois extension of local fields. Let $x \in L$ have conjugates $x_1 (= x), x_2, \ldots, x_n$ over $K$. Suppose $y \in L$ satisfies $|y - x| < |y - x_i|$ for $i \geq 2$. Show that $x \in K(y)$.

(b) Let $K$ be a local field and $f \in K[X]$ be a separable, irreducible polynomial of degree $n$, defining extension $L$ of $K$ (i.e. $L \cong K[X]/(f)$). Show that every polynomial $h \in K[X]$ of degree $n$ that is close enough to $f$, is irreducible and the extension $K[X]/(h)$ of $K$ is isomorphic to $L$.

2. (a) Construct an abelian extension of $\mathbb{Q}(\sqrt{2})$ that is not cyclotomic.

(b) (Kronecker-Weber for cubic extensions) Let $F$ be a Galois extension of $\mathbb{Q}$ of degree 3. Let $K = \mathbb{Q}(\zeta_3), L = F(\zeta_3)$, and $\text{Gal}(K/\mathbb{Q}) = \{1, \sigma\}$.

(i) Show that $L = K(\alpha^{1/3})$ for some cubefree $\alpha \in \mathbb{Z}[\zeta_3]$.

(ii) Show that $\sigma(\alpha) \equiv \alpha^2 \pmod{K^*3}$.

(iii) Show that if the rational prime $p$ ramifies in $F$, then $p = 3$ or $p \equiv 1 \pmod{3}$.

(iv) Show that $F/\mathbb{Q}$ is cyclotomic.