MATH 845: HOMEWORK 1, DUE FEB 10

1. (a) Let L/K be a finite Galois extension of local fields. Let $x \in L$ have conjugates $x_1(= x), x_2, ..., x_n$ over K. Suppose $y \in L$ satisfies $|y - x| < |y - x_i|$ for $i \geq 2$. Show that $x \in K(y)$.

(b) Let K be a local field and $f \in K[X]$ be a separable, irreducible polynomial of degree n, defining extension L of K (i.e. $L \cong K[X]/(f)$). Show that every polynomial $h \in K[X]$ of degree n that is close enough to f, is irreducible and the extension $K[X]/(h)$ of K is isomorphic to L.

2. (a) Construct an abelian extension of $\mathbf{Q}(\sqrt{2})$ that is not cyclotomic.

(b) (Kronecker-Weber for cubic extensions) Let F be a Galois extension of $\mathbf{Q}$ of degree 3. Let $K = \mathbf{Q}(\zeta_3), L = F(\zeta_3)$, and $\mathrm{Gal}(K/\mathbf{Q}) = \{1, \sigma\}$.

(i) Show that $L = K(\alpha^{1/3})$ for some cubefree $\alpha \in \mathbf{Z}[\zeta_3]$.

(ii) Show that $\sigma(\alpha) \equiv \alpha^2 \pmod{K^{*3}}$.

(iii) Show that if the rational prime p ramifies in F, then $p = 3$ or $p \equiv 1 \pmod 3$.

(iv) Show that $F/\mathbf{Q}$ is cyclotomic.