MATH 845: HOMEWORK 2, DUE FEB 24

1. (a) Let $f : \mathbf{Q}_p \to \mathbf{Q}_q$ be an isomorphism and a homeomorphism. Show that $p = q$.

(b) Let L be a field with a discrete valuation v. Let A be the set of all Cauchy sequences (α_n) $(\alpha_n \in L)$ and M the subset of those with $\alpha_n \to 0$ as $n \to \infty$. Show that A is naturally a ring with maximal ideal M and that $\hat{L} = A/M$ is a field containing L and possessing a discrete valuation restricting to v on L.

Let $\Updownarrow$ denote the elements of L with positive valuation (including 0). The $\Updownarrow$-adic topology on L is defined by taking the sets $\alpha + \Updownarrow^n$ $(n \geq 0)$ as open neighborhoods of $\alpha \in L$. Show that the completion of L in this $\Updownarrow$-adic topology coincides with $\hat{L}$.

2. (a) Let A be a commutative ring with 1 in which 2 is invertible. Show that

$$F_\alpha(X,Y) = \frac{X\sqrt{(1-Y^2)(1-\alpha^2Y^2)} + Y\sqrt{(1-X^2)(1-\alpha^2X^2)}}{1+\alpha^2X^2Y^2}$$

with $\alpha \in A$, defines a formal group over A.

(b) Let $F(X,Y) \in \mathbf{Z}[X,Y]$. Show that F defines a formal group over $\mathbf{Z}$ if and only if $F(X,Y) = X + Y + \alpha XY$ for some $\alpha \in \mathbf{Z}$.