# MATH 845: HOMEWORK 3, DUE MAR 17.

Let $G$ be a group acting by automorphisms on another (not necessarily abelian) group $M$. Define $H^0(G,M)$ to be $M^G$ as usual. A 1-cocycle is a map $f: G \to M$ such that $f(\sigma\tau) = f(\sigma)\sigma f(\tau)$ holds for all $\sigma, \tau \in G$. Two cocycles $f, g$ are called equivalent if there exists $m \in M$ such that $g(\sigma) = m^{-1}f(\sigma)\sigma(m)$ for all $\sigma \in G$. Then $H^1(G,M)$ is the set defined as 1-cocycles modulo equivalence. Note that it contains a distinguished element corresponding to the cocycle that sends every element of $G$ to the identity of $M$.

1. (a) Show that "equivalence" really is an equivalence relation, and that if $\sigma \mapsto f(\sigma)$ is a cocycle, then so is $\sigma \mapsto m^{-1}f(\sigma)\sigma(m)$ for every $m \in M$.

(b) Suppose $1 \to M' \to M \to M'' \to 1$ is a short exact sequence of groups with $G$-action. Show that the induced $1 \to M'^G \to M^G \to M''^G$ is exact.

(c) Construct a map $\delta: M''^G \to H^1(G,M')$ such that the image of $M^G$ consists of those elements mapping under $\delta$ to the distinguished element of $H^1(G,M')$ and such that the image of $\delta$ consists of those elements of $H^1(G,M')$ mapping to the distinguished element of $H^1(G,M)$.

2. (a) Let $L/K$ be a finite Galois extension with Galois group $G$ and $f: G \to GL_n(L)$ be a cocycle. Show that $f(\sigma) = A\sigma(A^{-1})$ if and only if each of the column vectors $v$ of $A$ satisfies the condition $v = f(\sigma)\sigma(v)$ for every $\sigma \in G$.

(b) Construct for each $w \in L^n$ a vector $v(w) \in L^n$ such that $v(w) = f(\sigma)\sigma(v(w))$ for every $\sigma \in G$.

(c) Show that the $L$-span of $\{v(w) : w \in L^n\}$ is all of $L^n$. Find $H^1(G, GL_n(L))$.