MATH 845: HOMEWORK 1, DUE FEB 12

1. Suppose that (a, b) is a rational point on Reichardt's curve $x^4 - 17 = 2y^2$.

(a) Show that $a = A/C, b = B/C^2$, where $A, B, C \in \mathbf{Z}$, $\gcd(A, B) = \gcd(A, C) = \gcd(B, C) = 1$.

(b) Rewriting the equation as $(5A^2 + 17C^2)^2 - (4B)^2 = 17(A^2 + 5C^2)^2$, show that there are $U, V \in \mathbf{Z}$ such that

$5A^2 + 17C^2 \pm 4B = 17U^2, 5A^2 + 17C^2 \mp 4B = V^2, A^2 + 5C^2 = UV$

or

$5A^2 + 17C^2 \pm 4B = 34U^2, 5A^2 + 17C^2 \mp 4B = 2V^2, A^2 + 5C^2 = 2UV$

(c) Hence show that (eliminating B) either

$10A^2 + 34B^2 = 17U^2 + V^2, A^2 + 5C^2 = UV$

or

$5A^2 + 17C^2 = 17U^2 + V^2, A^2 + 5C^2 = 2UV$

(d) Show that neither possibility has a solution modulo 17, and so deduce that Reichardt's equation has no rational solutions.

(e) Show, however, that Reichardt's equation has a real solution and a solution modulo p for each prime number $p \equiv 3 \pmod 4$. (It's possible but harder to show this for all primes p.)

2. Let E be an elliptic curve over an algebraically closed field of characteristic zero.

(a) Show that E has 9 points of inflection, which have the property that every line joining two of them contains a third.

(b) Show that by properly choosing the coordinate system any such set of points can be given the coordinates $(0, 1, -1), (-1, 0, 1), (1, -1, 0), (0, 1, \alpha), (\alpha, 0, 1), (1, \alpha, 0), (0, 1, \beta), (\beta, 0, 1), (1, \beta, 0)$ where α and β are the two roots of $x^2 - x + 1 = 0$.

(c) Show that any cubic curve passing through the above nine points necessarily has the form

$x^3 + y^3 + z^3 + 3mxyz = 0$ or $xyz = 0$ (this case corresponds to $m = \infty$).

(d) Show that this cubic is smooth if and only if $m \neq \infty, -1, \alpha, \beta$.