MATH 845: HOMEWORK 2, DUE FEB 26

3. Consider the problem of finding triples of positive integers (a, b, c) with no common factor such that the triangle ABC with sides a, b, c has the following property:

($*$) The median from A, the angle bisector from B, and the altitude from C meet in one point.

An example is $(12, 13, 15)$.

Setting $Y = 2b/(a+c), X = 2c/(a+c)$, ($*$) gives $Y^2 = X^3 - 4X + 4$.

(a) Sketch the graph of the curve for $-3 \leq X \leq 3$. Is it an elliptic curve?

(b) Show that a point (X, Y) on this curve for which there exists a corresponding triangle ABC, must satisfy $0 < X < 2, 0 < Y < 2$.

(c) Call the curve E. Find $\{P \in E(\mathbf{Q}) \mid 6P = \infty\}$. [Hint: think geometrically.]

(d) There is a particularly simple triangle satisfying ($*$), namely $(1, 1, 1)$. Show, however, that the corresponding point on E actually equals $-4Q$ where Q is the point $(2, 2)$.

(e) Show that $7Q$ also produces a solution to ($*$).

(f) Assuming $\phi : E(\mathbf{R}) \cong \mathbf{R}/\mathbf{Z}$ as topological groups (shown in class), what property of $\phi(Q)$ is enough to ensure that multiples of Q produce infinitely many noncongruent triangles satisfying ($*$)?

4. (a) Suppose E is the elliptic curve $y^2 = x^3 - n^2 x$ (n a squarefree integer), defined over $\mathbf{Q}$. Give an explicit expression for the x-coordinate of $2P$ in terms of the coordinates of P.

(b) Let X, Y, Z be the sides of a rational right triangle with area n. Let P be the corresponding point on E constructed in class. Show that $P = 2Q$ for some point in $E(\mathbf{Q})$.

(c) Let $P \in E(\mathbf{Q})$ not be in $E[2]$. Show that the x-coordinate of $2P$ is the square of a rational number having even denominator.

(d) Show that if $(x, y) \in E(\mathbf{Q})$ has x the square of a rational number with even denominator, then there exists a rational right triangle with area n.

(e) Deduce that n is congruent if and only if $E(\mathbf{Q}) \neq E[2]$.