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Integral invariants for surfaces in $\mathbb{R}^3$

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Abstract

There is a long history of differential invariants of Lie groups acting on manifolds. These are, however, not robust in the presence of noise, and so for object recognition applications an analogous theory of integral invariants was recently proposed. This paper presents the first systematic method for producing integral invariants for surfaces in $\mathbb{R}^3$.

1. INTRODUCTION

In the field of computer vision, the importance of invariants has been recognized since its origin in 1960s. The emphasis was initially on *photometric* invariants and this lead to the development of edge detectors. In this paper, we focus on *geometric* invariants, which are geometric properties remaining unchanged under a particular class of transformations.

Existing geometric invariants can be categorized into three types: algebraic, differential and integral invariants. Algebraic invariants were applied in recognizing industrial objects [3]. They are obtained by fitting polynomials to the shape and then calculating invariants from the polynomial coefficients. The algebraic approach has several problems: first, most shapes cannot be expressed in terms of simple polynomials, making the curve fitting difficult. Second, algebraic invariants are a global method because it requires the whole shape. Hence, they will not work under partial occlusion.

Differential invariants obtained by the *method of moving frames* have been applied in many computer vision problems [1], [2]. The fundamental problem of differential invariants is that they depend on high order derivatives, and thus are particularly sensitive to noise and round-off error. In order to overcome the limitations of differential invariants, there have been attempts to derive invariants based on integrals [4], [5], [6], which are called integral invariants. The previous works on integral invariants are all limited to the transformation groups acting on $\mathbb{R}^2$.

The problem of recognizing objects in 3D scenes has been intensively studied due to its importance in many applications. In this paper, the integral invariants of Lie groups acting on $\mathbb{R}^3$ are found. These invariants can be used to recognize 3D objects.

The rest of this paper is organized as follows. First, we briefly review the method of moving frames. Then for the Euclidean and affine groups acting on $\mathbb{R}^3$, integral invariants are explicitly derived.

2. INTEGRAL INVARIANTS

Hann and Hickman [4] were the first to derive integral invariants by prolonging a Lie group action to potentials. However, their approach does not generalize to the 3-dimensional case. For Lie groups acting on $\mathbb{R}^3$, their approach will produce complicated normalization equations whose associated moving frame cannot be easily solved. In the following sections, we will review the method of moving frames and show how integral invariants for Lie groups acting on $\mathbb{R}^3$ can be derived.

A. The Method of Moving Frames

The method of moving frames, introduced by Élie Cartan, is a powerful tool for finding invariants under group actions. Here, we present the basics of the moving frame method. Throughout this paper, G will denote an r-dimensional Lie group acting on an m-dimensional manifold M.

Definition 2.1. *An invariant of M under the action of G is a function $\eta : M \mapsto \mathbb{R}$ which satisfies $\eta(g \circ x) = \eta(x)$, for all $g \in G, x \in M$.*

Definition 2.2. *A moving frame is a smooth, G-equivariant map $\rho : M \mapsto G$, i.e. $\rho(g \circ x) = g\rho(x)$.*

The following theorem provides necessary and sufficient conditions for the existence of a moving frame.

Theorem 2.1. *A moving frame exits in a neighborhood of a point $x \in M$ if and only if G acts freely and regularly near x.*

Note that this implies that $r \leq m$. The construction of a moving frame is based on solving *normalization equations*, which require the choice of a set K of canonical forms, consisting of one fixed point in each orbit $O_x := \{g \circ x : g \in G\}$. Choosing coordinates $x = (x_1, \dots, x_m)$ of M such that $K = \{(x_1, \dots, x_r) : x_1 = c_1, \dots, x_r = c_r\}$, invariants are obtained as follows from the remaining $m - r$ coordinates.

Note that given a moving frame ρ, for each $x \in M$ $\rho(x)^{-1} \circ x \in O_x$ and for every $g \in G$ $\rho(g \circ x)^{-1} \circ (g \circ x) = \rho(x)^{-1} \circ x$. Thus, $K = \{\rho(x)^{-1} \circ x : x \in M\}$ is a set of canonical forms. Thus if we define $w_1, ..., w_m$ by $g^{-1} \circ x = (w_1(g, x), \dots, w_m(g, x))$, then $w_i(\rho(x), x) = c_i$ if $1 \leq i \leq r$. Second, note that since $\rho(g \circ x)^{-1} \circ (g \circ x) = \rho(x)^{-1} \circ x$, $w_i(\rho(g \circ x), g \circ x) = w_i(\rho(x), x)$ for all $x \in M, g \in G$, i.e. the $w_i(\rho(x), x)$ are invariants. In fact:

Theorem 2.2. *If $g^{-1} \circ x = (w_1(g, x), \dots, w_m(g, x))$, then $w_{r+1}(\rho(x), x), \dots, w_m(\rho(x), x)$ gives a complete list of functionally independent invariants of M under the action of group G.*

Unfortunately, in many object recognition problems, the dimension of the transformation group is greater than or equal to the dimension of the manifold on which it acts, i.e. $r > m$. This issue can be overcome by prolonging the group action to jet space. Traditionally, jet space is defined by using derivatives as new coordinates. The invariants obtained by prolonging the group action to derivatives are called *differential invariants*.

Invariants can also be obtained by prolonging the group action to integrals [4]. The term integral invariant is used to denote an invariant under the prolonged action. In [4], they derive integral invariants for curves in $\mathbb{R}^2$. In the next section, we will derive integral invariants for surfaces in $\mathbb{R}^3$. For space curves, integral invariants can be derived in the same manner.

B. Integral Invariants for Surfaces

Let us consider a surface $S \subseteq \mathbb{R}^3$ as a mapping from a region $[u_0, u_1] \times [v_0, v_1]$ to $\mathbb{R}^3$

$$S : (u, v) \mapsto (x, y, z), \text{where } (u, v) \in [u_0, u_1] \times [v_0, v_1] \tag{1}$$

Potentials of a surface are defined as below.

Definition 2.3. *The* ***potential*** $P_{i,j,k}$ *of order* n *is given by*

$$P_{i,j,k} = \int_{u_0}^{u_1} \int_{v_0}^{v_1} x^i \cdot y^j \cdot z^k \,\mathrm{d}u\,\mathrm{d}v \tag{2}$$

, where i, j, k *are nonnegative integers,* $i + j + k = n$ *and* $n \neq 0$*.*

Enlarging M by including potentials and prolonging the action of G, we next use the theory of moving frames to obtain integral invariants.

Theorem 2.3. *For a surface S, its integral invariants* $\eta_{i,j,k}$ *under Euclidean transformation can be expressed in compact form*

$$\eta_{i,j,k} = \int_{u_0}^{u_1} \int_{v_0}^{v_1} \bar{x}^i \bar{y}^j \bar{z}^k du dv$$

where i, j, k *are nonnegative integers,* $i + j + k \neq 0$ *and*

$$\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \mathbf{T})$$

The matrices $\mathbf{R}_1$*,* $\mathbf{R}_2$*,* $\mathbf{R}_3$ *and*

$\mathbf{T}$ *make up the inverse of a moving frame. The general form of* $\mathbf{R}_1$*,*

$\mathbf{R}_2$*,* $\mathbf{R}_3$ *and* $\mathbf{T}$ *are given by*

$$\mathbf{T} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \mathbf{R}_1 = \begin{bmatrix} \cos\omega_1 & \sin\omega_1 & 0 \\ -\sin\omega_1 & \cos\omega_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{R}_2 = \begin{bmatrix} \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 1 & 0 \\ \sin\omega_2 & 0 & \cos\omega_2 \end{bmatrix}, \mathbf{R}_3 = \begin{bmatrix} \cos\omega_3 & \sin\omega_3 & 0 \\ -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof. The Euclidean group acting on $\mathbb{R}^3$ has 5 dimensions. We can choose a moving frame so that its inverse moves $(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$ to the origin. Hence, we have

$$\mathbf{T} = \begin{bmatrix} x(u_0, v_0) \\ y(u_0, v_0) \\ z(u_0, v_0) \end{bmatrix}$$

Then we need to fix the other two degrees of freedom. Let

$$\mathbf{A} = \begin{bmatrix} x(u_1, v_0) \\ y(u_1, v_0) \\ z(u_1, v_0) \end{bmatrix} - \begin{bmatrix} x(u_0, v_0) \\ y(u_0, v_0) \\ z(u_0, v_0) \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} x(u_0, v_1) \\ y(u_0, v_1) \\ z(u_0, v_1) \end{bmatrix} - \begin{bmatrix} x(u_0, v_0) \\ y(u_0, v_0) \\ z(u_0, v_0) \end{bmatrix}$$

and $\mathbf{C} = \mathbf{A} \times \mathbf{B}$. A point in Cartesian coordinates (x, y, z) can be transformed into spherical coordinates (R, θ, ϕ). θ and ϕ are angular displacements measured from the positive z-axis and positive x-axis respectively and R is the distance from the origin to the point. The vector $\mathbf{C}$ is expressed in spherical coordinates (R_c, θ_c, ϕ_c). By setting $\omega_1 = \phi_c$ and $\omega_2 = \theta_c$, the rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ will rotate $\mathbf{A}$ and $\mathbf{B}$ so that they are lying in the x-y plane.

Now there is one degree of freedom left to be fixed. Let

$$\mathbf{D} = \mathbf{R}_2\mathbf{R}_1(\begin{bmatrix} x(u_1, v_0) \\ y(u_1, v_0) \\ z(u_1, v_0) \end{bmatrix} - \mathbf{T})$$

Again, the vector $\mathbf{D}$ can be expressed in spherical coordinates (R_D, θ_D, ϕ_D). By setting $\omega_3 = \phi_D$, the rotation matrices $\mathbf{R}_1$, $\mathbf{R}_2$, $\mathbf{R}_3$ and translation $\mathbf{T}$ form the inverse of a moving frame. We can construct integral invariants $\eta_{i,j,k}$ under Euclidean transformation by applying the inverse of the moving frame to the potential $P_{i,j,k}$. ☐

Theorem 2.4. *An integral invariant for the group $A(3)$ is given by*

$$\begin{aligned}
\eta_{2,0,0} =& \{P_{0,0,2}(A(x_{01}y_{00} - x_{00}y_{01}) + P_{100}(y_{01} - y_{00}) + P_{010}(x_{00} - x_{01}))^2 \\
& - 2P_{011}(A(x_{01}y_{00} - x_{00}y_{01}) + P_{100}(y_{01} - y_{00}) + P_{010}(x_{00} - x_{01})) \\
& (A(x_{01}z_{00} - x_{00}z_{01}) + P_{100}(z_{01} - z_{00}) + P_{001}(x_{00} - x_{01})) \\
& + P_{020}(A(x_{01}z_{00} - x_{00}z_{01}) + P_{100}(z_{01} - z_{00}) + P_{001}(x_{00} - x_{01}))^2 \\
& - 2P_{110}(A(x_{01}z_{00} - x_{00}z_{01}) + P_{100}(z_{01} - z_{00}) + P_{001}(x_{00} - x_{01})) \\
& (A(y_{01}z_{00} - y_{00}z_{01}) + P_{010}(z_{01} - z_{00}) + P_{001}(y_{00} - y_{01})) \\
& + P_{200}(A(y_{01}z_{00} - y_{00}z_{01}) + P_{010}(z_{01} - z_{00}) + P_{001}(y_{00} - y_{01}))^2 \\
& - 2P_{101}(A(x_{01}y_{00} - x_{00}y_{01}) + P_{100}(y_{01} - y_{00}) + P_{010}(x_{00} - x_{01})) \\
& (A(y_{00}z_{01} - y_{01}z_{00}) + P_{010}(z_{00} - z_{01}) + P_{001}(y_{01} - y_{00})) \\
& - A(P_{100}(y_{01}z_{00} - y_{00}z_{01}) + P_{010}(x_{00}z_{01} - x_{01}z_{00}) + P_{001}(x_{01}y_{00} - x_{00}y_{01}))^2\} \\
& /(A(x_{00}(y_{10}z_{01} - y_{01}z_{10}) + x_{10}(y_{01}z_{00} - y_{00}z_{01}) + x_{01}(y_{00}z_{10} - y_{10}z_{00})) \\
& + P_{100}(y_{00}(z_{01} - z_{10}) + y_{10}(z_{00} - z_{01}) + y_{01}(z_{10} - z_{00})) \\
& + P_{010}(x_{00}(z_{10} - z_{01}) + x_{10}(z_{01} - z_{00}) + x_{01}(z_{00} - z_{10})) \\
& + P_{001}(x_{00}(y_{01} - y_{10}) + x_{10}(y_{00} - y_{01}) + x_{01}(y_{10} - y_{00})))^2
\end{aligned} \tag{3}$$

, where $A = (u_1 - u_0)(v_1 - v_0)$, $x_{00} = x(u_0, v_0)$, $y_{00} = y(u_0, v_0)$, $x_{01} = x(u_0, v_1)$, $y_{01} = y(u_0, v_1)$, $x_{10} = x(u_1, v_0)$ and $y_{10} = y(u_1, v_0)$.

Proof. The action of the affine group $G := \{(u,v)|u \in GL_3(\mathbb{R}), v \in \mathbb{R}^3\}$ on $\mathbb{R}^3$ can be described as below,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} j \\ k \\ \ell \end{bmatrix} = \begin{bmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{bmatrix}$$

In order to find invariants, we need to create a potential jet space whose dimension is greater than the dimension of the affine group. In this case, the group has dimension 12. Here a potential jet space with dimension 13, shown below, is used to obtain an integral invariant.

$$(x(u_0,v_0), y(u_0,v_0), z(u_0,v_0), x(u_1,v_0), y(u_1,v_0), z(u_1,v_0),$$
$$x(u_0,v_1), y(u_0,v_1), z(u_0,v_1), P_{1,0,0}, P_{0,1,0}, P_{0,0,1}, P_{2,0,0})$$

The first 12 coordinates in potential jet space will be used to construct a moving frame. In order to simplify the algebraic computation, we choose the fixed coordinates to be $(0,0,0,1,0,0,0,1,0,0,0,0)$. The equations below

$$\begin{array}{ccc} \overline{x}(u_0,v_0) = 0 & \overline{y}(u_0,v_0) = 0 & \overline{z}(u_0,v_0) = 0 \\ \overline{x}(u_1,v_0) = 1 & \overline{y}(u_1,v_0) = 0 & \overline{z}(u_1,v_0) = 0 \\ \overline{x}(u_0,v_1) = 0 & \overline{y}(u_0,v_1) = 1 & \overline{z}(u_0,v_1) = 0 \\ \overline{P}_{1,0,0} = 0 & \overline{P}_{0,1,0} = 0 & \overline{P}_{0,0,1} = 0 \end{array}$$

are the *normalization equations* where $\overline{P}_{1,0,0}$ is given by

$$\begin{aligned} \overline{P}_{1,0,0} &= \int_{u_0}^{u_1} \int_{v_0}^{v_1} \overline{x}(u,v)dudv \\ &= \int_{u_0}^{u_1} \int_{v_0}^{v_1} (ax+by+cz+j)dudv \\ &= aP_{1,0,0} + bP_{0,1,0} + cP_{0,0,1} + j(u_1-u_0)(v_1-v_0) \end{aligned}$$

Similarly,

$$\begin{aligned} \overline{P}_{0,1,0} &= \int_{u_0}^{u_1} \int_{v_0}^{v_1} \overline{y}(u,v)dudv \\ &= \int_{u_0}^{u_1} \int_{v_0}^{v_1} (dx+ey+fz+k)dudv \\ &= dP_{1,0,0} + eP_{0,1,0} + fP_{0,0,1} + k(u_1-u_0)(v_1-v_0) \end{aligned}$$

$$\begin{aligned} \overline{P}_{0,0,1} &= \int_{u_0}^{u_1} \int_{v_0}^{v_1} \overline{z}(u,v)dudv \\ &= \int_{u_0}^{u_1} \int_{v_0}^{v_1} (gx+hy+iz+\ell)dudv \\ &= gP_{1,0,0} + hP_{0,1,0} + iP_{0,0,1} + \ell(u_1-u_0)(v_1-v_0) \end{aligned}$$

By solving the normalization equations, we have the moving frame. The fundamental invariant is found by applying the inverse of the moving frame to the thirteenth coordinate. Furthermore, infinitely many integral invariants could be found by applying the inverse of the moving frame to higher order potentials. Here, we use $P_{2,0,0}$ as an example. The affine-transformed potential $\overline{P}_{2,0,0}$ is given by

$$\begin{aligned}\overline{P}_{2,0,0} &= \int_{u_0}^{u_1}\int_{v_0}^{v_1}\overline{x}^2(u,v)dudv \\ &= a^2P_{2,0,0}+b^2P_{0,2,0}+c^2P_{0,0,2}+2j(aP_{1,0,0}+bP_{0,1,0}+cP_{0,0,1}) \\ &\quad +2abP_{1,1,0}+2bcP_{0,1,1}+2acP_{1,0,1}+j^2(u_1-u_0)(v_1-v_0)\end{aligned}$$

By substituting the inverse of the moving frame $\{a,b,c,d,e,f,g,h,i,j,k,\ell\}$ into $\overline{P}_{2,0,0}$, we have the integral invariant $\eta_{2,0,0}$ which is shown in equation 3. $\square$

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